

A MAPPING ASSOCIATED WITH JENSEN'S INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper we introduce a new mapping connected with the classical inequality due to Jensen and point out its main properties. Some applications related to well-known inequalities are also established.

1. INTRODUCTION

Let X be a real linear space and C a convex subset in X , where $f : C \rightarrow \mathbb{R}$ is a convex function. Suppose that $x_i \in C$ ($i \in I$), $p_i \geq 0$, with $P_I := \sum_{i \in I} p_i > 0$, where $I \subset \mathbb{N}$ is a finite set of indices. The following inequality is well-known in literature as Jensen's discrete inequality:

$$(1.1) \quad f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \leq \frac{1}{P_I} \sum_{i \in I} p_i f(x_i).$$

Note that some of the classical inequalities (the arithmetic mean-geometric mean inequality, Levinson's inequality, Ky Fan's inequality, etc) are particular cases of this inequality (see also [6] and [8] where further references are given). For some results which give refinements, counterparts and inequalities related to Jensen's inequality (1.1), we refer the reader to the papers [1]–[5] and [7].

We introduce the following notations (see also [5]):

$$\begin{aligned} \mathfrak{P}_f(\mathbb{N}) &:= \{I \subset \mathbb{N} \mid I \text{ is finite}\} \\ \mathfrak{J}^+(\mathbb{R}) &:= \{p = (p_i)_{i \in \mathbb{N}} \mid p_i > 0 \text{ for all } i \in \mathbb{N}\} \\ \mathfrak{J}_*(C) &:= \{x = (x_i)_{i \in \mathbb{N}} \mid x_i \in C \text{ for all } i \in \mathbb{N}\} \\ \text{Conv}(C, \mathbb{R}) &:= \text{the cone of all convex mappings defined on } C. \end{aligned}$$

In what follows, we consider the map $H : \text{Conv}(C, \mathbb{R}) \times \mathfrak{P}_f(\mathbb{N}) \times \mathfrak{J}^+(\mathbb{R}) \times \mathfrak{J}_*(C) \rightarrow \mathbb{R}_+$ and given by

$$H = H(f, I, p, x) := \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \right]^{P_I} \geq 0.$$

Further on, we shall point out some properties for the mapping H which improve inequality (1.1). Some particular examples are also examined.

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2. THE RESULTS

We will start with the following result.

Theorem 1. *Suppose that f, p and x are as above. Then for all I, J finite sets of indices with $I \cap J = \emptyset$ one has the inequality*

$$(2.1) \quad H(f, I \cup J, p, x) \geq H(f, I, p, x) \cdot H(f, J, p, x) \geq 0$$

i.e., the mapping $H(f, \cdot, p, x)$ is supermultiplicative on $\mathfrak{P}_f(\mathbb{N})$.

Proof. Consider the mapping $L : \text{Conv}(C, \mathbb{R}) \times \mathfrak{P}_f(\mathbb{N}) \times \mathfrak{J}^+(\mathbb{R}) \times \mathfrak{J}_*(C) \rightarrow \mathbb{R}_+$ given by

$$L(f, I, p, x) := \frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \geq 0.$$

For all $I, J \in \mathfrak{P}_f(\mathbb{N})$ with $I \cap J = \emptyset$ one has:

$$\begin{aligned} L(f, I \cup J, p, x) &= \frac{1}{P_I + P_J} \left(\sum_{i \in I} p_i f(x_i) + \sum_{j \in J} p_j f(x_j) \right) \\ &\quad - f\left(\frac{1}{P_I + P_J} \left(\sum_{i \in I} p_i x_i + \sum_{j \in J} p_j x_j \right)\right). \end{aligned}$$

By the convexity of f one has

$$\begin{aligned} &f\left(\frac{P_I}{P_I + P_J} \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + \frac{P_J}{P_I + P_J} \left(\frac{1}{P_J} \sum_{j \in J} p_j x_j\right)\right) \\ &\leq \frac{P_I}{P_I + P_J} f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + \frac{P_J}{P_I + P_J} f\left(\frac{1}{P_J} \sum_{j \in J} p_j x_j\right) \end{aligned}$$

and thus

$$\begin{aligned} L(f, I \cup J, p, x) &\geq \frac{1}{P_I + P_J} \left[\sum_{i \in I} p_i f(x_i) + \sum_{j \in J} p_j f(x_j) \right] \\ &\quad - \frac{P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + P_J f\left(\frac{1}{P_J} \sum_{j \in J} p_j x_j\right)}{P_I + P_J} \\ &= \frac{P_I L(f, I, p, x) + P_J L(f, J, p, x)}{P_I + P_J}. \end{aligned}$$

Using the elementary arithmetic mean-geometric mean inequality:

$$\frac{\alpha a + \beta b}{\alpha + \beta} \geq a^{\frac{\alpha}{\alpha + \beta}} b^{\frac{\beta}{\alpha + \beta}} \quad \text{with } a, b \geq 0, \alpha, \beta \geq 0 \text{ with } \alpha + \beta > 0,$$

we obtain:

$$L(f, I \cup J, p, x) \geq [L(f, I, p, x)]^{\frac{P_I}{P_I + P_J}} [L(f, J, p, x)]^{\frac{P_J}{P_I + P_J}}$$

that is,

$$(2.2) \quad [L(f, I \cup J, p, x)]^{P_I \cup J} \geq [L(f, I, p, x)]^{P_I} [L(f, J, p, x)]^{P_J}$$

and the inequality (2.1) is obtained. ■

The following corollary also holds.

Corollary 1. *Let H_0 be a fixed set in $\mathfrak{P}_f(\mathbb{N})$ with $L(f, H_0, p, x) > 0$. Then for all $I, J \in \mathfrak{P}_f(\mathbb{N})$ with $I \supset J$ and $I \setminus J = H_0$, one has the inequality*

$$(2.3) \quad \left[\frac{L(f, I, p, x)}{L(f, H_0, p, x)} \right]^{P_I} \geq \left[\frac{L(f, J, p, x)}{L(f, H_0, p, x)} \right]^{P_J}.$$

Proof. Using the inequality (2.2), we can write

$$\begin{aligned} [L(f, I, p, x)]^{P_I} &= [L(f, H_0 \cup J, p, x)]^{P_{H_0 \cup J}} \\ &\geq [L(f, H_0, p, x)]^{P_{H_0}} [L(f, J, p, x)]^{P_J} \\ &= [L(f, H_0, p, x)]^{P_I - P_J} [L(f, J, p, x)]^{P_J}, \end{aligned}$$

whence we obtain the inequality (2.3). ■

Another result for the mapping H defined above is given in the following theorem.

Theorem 2. *Suppose that $f : C \subseteq X \rightarrow \mathbb{R}$ is a convex function on the convex set C , $I \in \mathfrak{P}_f(\mathbb{N})$ and $x = (x_i)_{i \in \mathbb{N}} \in \mathfrak{J}_*(C)$. Then for all $p, q \in \mathfrak{J}^+(\mathbb{R})$, one has the inequality*

$$(2.4) \quad H(f, I, p + q, x) \geq H(f, I, p, x) H(f, I, q, x) \geq 0$$

that is, the mapping $H(f, I, \cdot, x)$ is supermultiplicative on $\mathfrak{J}^+(\mathbb{R})$.

Proof. As above, we have

$$\begin{aligned} &L(f, I, p + q, x) \\ &= \frac{1}{P_I + Q_I} \left[\sum_{i \in I} p_i f(x_i) + \sum_{i \in I} q_i f(x_i) \right] - f \left(\frac{1}{P_I + Q_I} \left(\sum_{i \in I} p_i x_i + \sum_{i \in I} q_i x_i \right) \right) \\ &\geq \frac{1}{P_I + Q_I} \left[\sum_{i \in I} p_i f(x_i) + \sum_{i \in I} q_i f(x_i) \right] \\ &\quad - \frac{P_I f \left(\sum_{i \in I} p_i x_i / P_I \right) + Q_I f \left(\sum_{i \in I} q_i x_i / Q_I \right)}{P_I + Q_I} \\ &= \frac{P_I L(f, I, p, x) + Q_I L(f, I, q, x)}{P_I + Q_I} \\ &\geq [L(f, I, p, x)]^{\frac{P_I}{P_I + Q_I}} [L(f, I, q, x)]^{\frac{Q_I}{P_I + Q_I}} \end{aligned}$$

whence we obtain

$$[L(f, I, p + q, x)]^{P_I + Q_I} \geq [L(f, I, p, x)]^{P_I} [L(f, I, q, x)]^{Q_I}$$

and the inequality (2.4) is proved. ■

The following corollary also holds.

Corollary 2. *Suppose that f, I, x are as above and $p, q \in \mathfrak{J}^+(\mathbb{R})$ such that $p - q = e \in \mathfrak{J}^+(\mathbb{R})$ with $L(f, I, e, x) > 0$. Then we have the inequality*

$$(2.5) \quad \left[\frac{L(f, I, p, x)}{L(f, I, e, x)} \right]^{P_I} \geq \left[\frac{L(f, I, q, x)}{L(f, I, e, x)} \right]^{Q_I}.$$

Proof. One has

$$\begin{aligned} [L(f, I, p, x)]^{P_I} &= [L(f, I, e + q, x)]^{E_I + Q_I} \\ &\geq [L(f, I, e, x)]^{E_I} [L(f, I, q, x)]^{Q_I} \\ &= [L(f, I, e, x)]^{P_I - Q_I} [L(f, I, q, x)]^{Q_I}, \end{aligned}$$

which proves the inequality (2.5). ■

Now, suppose that $q, e \in \mathfrak{J}^+(\mathbb{R})$ and $L(f, I, e, x) > 0$. We shall consider the mapping $Q : [0, \infty) \rightarrow [0, \infty)$ given by

$$Q(t) := \left[\frac{L(f, I, te + q, x)}{L(f, I, e, x)} \right]^{(tE_I + Q_I)}.$$

The main properties of this mapping are given in the following theorem.

Theorem 3. *With the above assumptions, one has*

- (i) *The mapping Q is logarithmically concave on $[0, \infty)$;*
- (ii) *The mapping Q is monotonic increasing on $[0, \infty)$;*
- (iii) *We have the bound*

$$(2.6) \quad \inf_{t \in [0, \infty)} Q(t) = Q(0) = \left[\frac{L(f, I, q, x)}{L(f, I, e, x)} \right]^{Q_I}.$$

Proof. The proof is as follows.

- (i) Let $t_1, t_2 \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. We have

$$\begin{aligned} &Q(\alpha t_1 + \beta t_2) \\ &= \left[\frac{L(f, I, (\alpha t_1 + \beta t_2)e + (\alpha + \beta)q, x)}{L(f, I, e, x)} \right]^{(\alpha t_1 + \beta t_2)E_I + (\alpha + \beta)Q_I} \\ &= \frac{H(f, I, \alpha(t_1e + q) + \beta(t_2e + q), x)}{[L(f, I, e, x)]^{\alpha(t_1E_I + Q_I) + \beta(t_2E_I + Q_I)}} \\ &\geq \frac{H(f, I, \alpha(t_1e + q), x) H(f, I, \beta(t_2e + q), x)}{[L(f, I, e, x)]^{\alpha(t_1E_I + Q_I)} [L(f, I, e, x)]^{\beta(t_2E_I + Q_I)}} \\ &= \left[\frac{L(f, I, \alpha(t_1e + q), x)}{L(f, I, e, x)} \right]^{\alpha(t_1E_I + Q_I)} \left[\frac{L(f, I, \beta(t_2e + q), x)}{L(f, I, e, x)} \right]^{\beta(t_2E_I + Q_I)} \\ &= \left\{ \left[\frac{L(f, I, t_1e + q, x)}{L(f, I, e, x)} \right]^{(t_1E_I + Q_I)} \right\}^\alpha \left\{ \left[\frac{L(f, I, t_2e + q, x)}{L(f, I, e, x)} \right]^{(t_2E_I + Q_I)} \right\}^\beta \\ &= [Q(t_1)]^\alpha [Q(t_2)]^\beta \end{aligned}$$

as for all $\delta > 0$, $s \in \mathfrak{J}^+(\mathbb{R})$, $L(f, I, \delta s, x) = L(f, I, s, x)$, and the logarithmic concavity of Q is proved.

(ii) Let $0 \leq t_1 < t_2 < \infty$. Then

$$\begin{aligned}
 Q(t_2) &= \left[\frac{L(f, I, (t_2 - t_1)e + t_1e + q, x)}{L(f, I, e, x)} \right]^{(t_2 - t_1)E_I + t_1E_I + Q_I} \\
 &= \frac{H(f, I, (t_2 - t_1)e + t_1e + q, x)}{[L(f, I, e, x)]^{(t_2 - t_1)E_I + t_1E_I + Q_I}} \\
 &\geq \frac{H(f, I, (t_2 - t_1)e, x) H(f, I, t_1e + q, x)}{[L(f, I, e, x)]^{(t_2 - t_1)E_I} [L(f, I, e, x)]^{t_1E_I + Q_I}} \\
 &= \left[\frac{L(f, I, (t_2 - t_1)e, x)}{L(f, I, e, x)} \right]^{(t_2 - t_1)E_I} \left[\frac{L(f, I, t_1e + q, x)}{L(f, I, e, x)} \right]^{t_1E_I + Q_I} \\
 &= \left[\frac{L(f, I, t_1e + q, x)}{L(f, I, e, x)} \right]^{t_1E_I + Q_I} = Q(t_1)
 \end{aligned}$$

as $L(f, I, (t_1 - t_2)e, x) = L(f, I, e, x)$; and the monotonicity of Q is proved.
 (iii) Now, since $Q(t) \geq Q(0)$ for all $t \in [0, \infty)$, the bound (2.6) is obtained.

■

3. APPLICATIONS

- Let $f : C \subseteq X \rightarrow \mathbb{R}$ be a convex mapping on the convex set C and $p_i \geq 0$, $i = \overline{1, 2n}$ with $\sum_{i=1}^{2n} p_i > 0$, $\sum_{i=1}^n p_{2i} > 0$, $\sum_{i=1}^n p_{2i-1} > 0$ and $x_i \in C$ ($i = \overline{1, 2n}$). Then we have the inequality:

$$\begin{aligned}
 &\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i f(x_i) - f\left(\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i x_i\right) \\
 &\geq \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i}\right) \right]^{\sum_{i=1}^n p_{2i} / P_{2n}} \\
 &\quad \times \left[\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} f(x_{2i-1}) - f\left(\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} x_{2i-1}\right) \right]^{\sum_{i=1}^n p_{2i-1} / P_{2n}} \\
 &\geq 0.
 \end{aligned}$$

- With the above assumptions and assuming that $p_i \geq 0$ ($i = \overline{1, 2n-1}$), with $\sum_{i=1}^{2n} p_i > 0$, $\sum_{i=1}^n p_{2i} > 0$, $\sum_{i=1}^n p_{2i-1} > 0$ and $x_i \in C$ ($i = \overline{1, 2n-1}$), one has the

inequality

$$\begin{aligned}
& \frac{1}{P_{2n-1}} \sum_{i=1}^{2n-1} p_i f(x_i) - f\left(\frac{1}{P_{2n-1}} \sum_{i=1}^{2n-1} p_i x_i\right) \\
& \geq \left[\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} f(x_{2i-1}) - f\left(\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} x_{2i-1}\right) \right]^{\sum_{i=1}^n p_{2i-1}/P_{2n-1}} \\
& \quad \times \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i}\right) \right]^{\sum_{i=1}^n p_{2i}/P_{2n-2}}.
\end{aligned}$$

3. Let $f : C \subseteq X \rightarrow \mathbb{R}$ be a convex mapping on C , $x_i \in C$ ($i = \overline{1, n}$) and $\alpha_i \in (0, \frac{\pi}{2})$, $i = \overline{1, n}$. Then we have the inequality

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\
& \geq \left[\frac{1}{\sum_{i=1}^n \sin^2 \alpha_i} \sum_{i=1}^n \sin^2 \alpha_i f(x_i) - f\left(\frac{\sum_{i=1}^n \sin^2 \alpha_i \cdot x_i}{\sum_{i=1}^n \sin^2 \alpha_i}\right) \right]^{\sum_{i=1}^n \sin^2 \alpha_i / n} \\
& \quad \times \left[\frac{1}{\sum_{i=1}^n \cos^2 \alpha_i} \sum_{i=1}^n \cos^2 \alpha_i f(x_i) - f\left(\frac{\sum_{i=1}^n \cos^2 \alpha_i \cdot x_i}{\sum_{i=1}^n \cos^2 \alpha_i}\right) \right]^{\sum_{i=1}^n \cos^2 \alpha_i / n} \\
& \geq 0.
\end{aligned}$$

4. Let X be a normed linear space and $p \geq 1$. Then for all $I, J \in \mathfrak{P}_f(\mathbb{N})$, where $I \cap J = \emptyset$, and $p_i \geq 0$ with $P_I, P_J > 0$, one has the inequality:

$$\begin{aligned}
& P_{I \cup J}^{p-1} \sum_{i \in I \cup J} p_i \|x_i\|^p - \left\| \sum_{i \in I \cup J} p_i x_i \right\|^p \\
& \geq \frac{P_{I \cup J}^p}{P_I^{p \cdot \frac{P_I}{P_{I \cup J}}} P_J^{p \cdot \frac{P_J}{P_{I \cup J}}}} \left(P_I^{p-1} \sum_{i \in I} p_i \|x_i\|^p - \left\| \sum_{i \in I} p_i x_i \right\|^p \right)^{\frac{P_I}{P_{I \cup J}}} \\
& \quad \times \left(P_J^{p-1} \sum_{j \in J} p_j \|x_j\|^p - \left\| \sum_{j \in J} p_j x_j \right\|^p \right)^{\frac{P_J}{P_{I \cup J}}} \\
& \geq 0
\end{aligned}$$

for all $x_i \in X$, $i \in I \cup J$.

If we assume that $p_i, q_i \geq 0$ so that $P_I, Q_I > 0$, $I \in \mathfrak{P}_f(\mathbb{N})$, then one has the

inequality:

$$\begin{aligned} & (P_I + Q_I)^{p-1} \sum_{i \in I} (p_i + q_i) \|x_i\|^p - \left\| \sum_{i \in I} (p_i + q_i) x_i \right\|^p \\ & \geq \frac{(P_I + Q_I)^p}{P_I^{p \frac{P_I}{P_I+Q_I}} \cdot Q_I^{q \frac{Q_I}{P_I+Q_I}}} \left(P_I^{p-1} \sum_{i \in I} p_i \|x_i\|^p - \left\| \sum_{i \in I} p_i x_i \right\|^p \right)^{\frac{P_I}{P_I+Q_I}} \\ & \quad \times \left(Q_I^{p-1} \sum_{i \in I} q_i \|x_i\|^p - \left\| \sum_{i \in I} q_i x_i \right\|^p \right)^{\frac{Q_I}{P_I+Q_I}} \end{aligned}$$

for all $x_i \in X$ ($i \in I$).

5. Now, let $x_i > 0$ and $p_i \geq 0$ ($i \in \mathbb{N}$) so that $P_I, P_J > 0$, $I, J \in \mathfrak{P}_f(\mathbb{N})$, $I \cap J = \emptyset$. Denote

$$A(I, p, x) := \frac{1}{P_I} \sum_{i \in I} p_i x_i$$

and

$$G(I, p, x) := \left(\prod_{i \in I} x_i^{p_i} \right)^{1/P_I}.$$

The following inequality is well-known in the literature as the arithmetic mean-geometric mean inequality:

$$(3.1) \quad A(I, p, x) \geq G(I, p, x).$$

By Theorem 1 applied for the mapping $f : (0, \infty) \rightarrow \mathbb{R}$, $f(x) = -\ln x$, we have:

$$\frac{A(I \cup J, p, x)}{G(I \cup J, p, x)} \geq \exp \left\{ \left[\ln \left(\frac{A(I, p, x)}{G(I, p, x)} \right) \right]^{\frac{P_I}{P_I+P_J}} \left[\ln \left(\frac{A(J, p, x)}{G(J, p, x)} \right) \right]^{\frac{P_J}{P_I+P_J}} \right\} \geq 1,$$

which gives a refinement of the well-known inequality (3.1).

If $p_i, q_i \geq 0$ such that $P_I, Q_I > 0$ and $x_i > 0$ ($i \in I$), then by Theorem 2 we have:

$$\frac{A(I, p+q, x)}{G(I, p+q, x)} \geq \exp \left\{ \left[\ln \left(\frac{A(I, p, x)}{G(I, p, x)} \right) \right]^{\frac{P_I}{P_I+Q_I}} \left[\ln \left(\frac{A(I, q, x)}{G(I, q, x)} \right) \right]^{\frac{Q_I}{P_I+Q_I}} \right\} \geq 1,$$

which also gives a refinement of (3.1).

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