

ON SOME GRÜSS TYPE INEQUALITY IN 2-INNER PRODUCT SPACES AND APPLICATIONS

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ABSTRACT. In this paper, we shall give a generalization of the Grüss type inequality and obtain some applications of the Grüss type inequality in terms of 2-inner product spaces.

1. Introduction

Let X be a linear space of dimension greater than 1 and $(\cdot, \cdot|\cdot)$ be a real-valued function on $X \times X \times X$ satisfying the following conditions:

- (2I₁) $(x, x|z) \geq 0$,
 $(x, x|z) = 0$ if and only if x and z are linearly dependent,
- (2I₂) $(x, x|z) = (z, z|x)$,
- (2I₃) $(x, y|z) = (y, x|z)$,
- (2I₄) $(\alpha x, y|z) = \alpha(x, y|z)$ for any real number α ,
- (2I₅) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a *2-inner product* and $(X, (\cdot, \cdot|\cdot))$ is called a *2-inner product space* (or a *2-pre-Hilbert space*) ([3]).

Some basic properties of the 2-inner product $(\cdot, \cdot|\cdot)$ are as follows ([3], [4]):

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(1) For all $x, y, z \in X$,

$$|(x, y|z)| \leq \sqrt{(x, x|z)}\sqrt{(y, y|z)}.$$

(2) For all $x, y \in X$, $(x, y|y) = 0$.

(3) If $(X, (\cdot, \cdot))$ is an inner product space, then the 2-inner product $(\cdot, \cdot| \cdot)$ is defined on X by

$$(x, y|z) = \begin{vmatrix} (x|y) & (x|z) \\ (y|z) & (z|z) \end{vmatrix} = (x|y)\|z\|^2 - (x|z)(y|z)$$

for all $x, y, z \in X$.

Under the same assumptions over X , the real-valued function $\|\cdot, \cdot\|$ on $X \times X$ satisfying the following conditions:

(2N₁) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(2N₂) $\|x, y\| = \|y, x\|$,

(2N₃) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all real number α ,

(2N₄) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\cdot, \cdot\|$ is called a *2-norm* on X and $(X, \|\cdot, \cdot\|)$ is called a *linear 2-normed space* ([7]).

Note that it is easy to show that the 2-norm $\|\cdot, \cdot\|$ is non-negative and, for all $x, y \in X$ and real numbers α , $\|x, y + \alpha x\| = \|x, y\|$.

For any non-zero x_1, x_2, \dots, x_n in X , let $V(x_1, x_2, \dots, x_n)$ denote the subspace of X generated by x_1, x_2, \dots, x_n . Whenever the notation $V(x_1, x_2, \dots, x_n)$ is used, by it will understood x_1, x_2, \dots, x_n to be linearly independent.

Note that, on any 2-inner product space $(X, (\cdot, \cdot| \cdot))$, $\|x, y\| = \sqrt{(x, x|y)}$ defines a 2-norm for which we have

$$(1.1) \quad (x, y|z) = \frac{1}{4}(\|x + y, z\|^2 - \|x - y, z\|^2),$$

$$(1.2) \quad \|x + y, z\|^2 + \|x - y, z\|^2 = 2(\|x, z\|^2 + \|y, z\|^2)$$

for all $x, y, z \in X$. On the other hand, if $(X, \|\cdot, \cdot\|)$ is a linear 2-normed space in which the condition (1.2) is satisfied for all $x, y, z \in X$, then we can define a 2-inner product $(\cdot, \cdot| \cdot)$ on X by the condition (1.1).

For a 2-inner product space $(X, (\cdot, \cdot | \cdot))$, Cauchy-Schwarz's inequality

$$(1.3) \quad |(x, y|z)| \leq (x, x|z)^{1/2}(y, y|z)^{1/2} = \|x, z\| \|y, z\|,$$

a 2-dimensional analogue of Cauchy-Schwarz's inequality, holds.

For further details on 2-inner product spaces and linear 2-normed spaces, refer to the papers ([2]-[5], [9], [10]).

Y. J. Cho et al. ([1]), S. S. Dragomir et al. ([6]) studied the inequalities of 2-inner product spaces and obtained some related results.

In this paper, we shall give a generalization of the Grüss type inequality and obtain some applications of the Grüss type inequality in terms of 2-inner product spaces.

2. The Main Results

In 1935, G. Grüss proved the integral inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_b^a f(x)g(x)dx - \frac{1}{b-a} \int_b^a f(x)dx \cdot \frac{1}{b-a} \int_b^a g(x)dx \right| \\ & \leq \frac{1}{4}(M-m)(N-n) \end{aligned}$$

if f and g are two integrable functions on $[a, b]$ satisfying the condition:

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N$$

for all $x \in [a, b]$ ([8]).

In this section, we shall give a generalization of the Grüss type inequality in terms of 2-inner product spaces.

Theorem 2.1. *Let $(X, (\cdot, \cdot | \cdot))$ be a 2-inner product space and $x, y, z, e \in X$ with $\|e, z\| = 1$ and $z \notin V(x, e, y)$. If m, n, M, N are real numbers such that*

$$(2.1) \quad (Me - x, x - me|z) \geq 0, \quad (Ne - y, y - ne|z) \geq 0,$$

then we have the inequality

$$(2.2) \quad |(x, y|z) - (x, e|z)(e, y|z)| \leq \frac{1}{4}|M - m||N - n|.$$

Proof. Note that

$$(x, y|z) - (x, e|z)(e, y|z) = (x - (x, e|z)e, y - (e, y|z)e|z).$$

By the Cauchy-Schwarz's inequality (1.3),

$$(2.3) \quad \begin{aligned} & |(x - (x, e|z)e, y - (e, y|z)e|z)|^2 \\ & \leq \|x - (x, e|z)e, z\|^2 \|y - (e, y|z)e, z\|^2 \\ & = (\|x, z\|^2 - |(x, e|z)|^2)(\|y, z\|^2 - |(e, y|z)|^2). \end{aligned}$$

On the other hand, we have

$$(2.4) \quad (M - (x, e|z))((x, e|z) - m) - (Me - x, x - me|z) = \|x, z\|^2 - |(x, e|z)|^2$$

and

$$(2.5) \quad (N - (e, y|z))((e, y|z) - n) - (Ne - y, y - ne|z) = \|y, z\|^2 - |(e, y|z)|^2.$$

Since $(Me - x, x - me|z) \geq 0$, $(Ne - y, y - ne|z) \geq 0$, we have

$$(2.6) \quad (M - (x, e|z))((x, e|z) - m) \geq \|x, z\|^2 - |(x, e|z)|^2$$

and

$$(2.7) \quad (N - (e, y|z))((e, y|z) - n) \geq \|y, z\|^2 - |(e, y|z)|^2.$$

Also, by the inequality $4ab \leq (a + b)^2$ for $a, b \in R$, we have

$$(2.8) \quad (M - (x, e|z))((x, e|z) - m) \leq \frac{1}{4}(M - m)^2$$

and, similarly,

$$(2.9) \quad (N - (e, y|z))((e, y|z) - n) \leq \frac{1}{4}(N - n)^2.$$

Thus, using (2.3)~(2.9), we have the inequality

$$|(x, y|z) - (x, e|z)(e, y|z)|^2 \leq \frac{1}{16}|M - m|^2|N - n|^2$$

and so we have the desired inequality (2.2). This completes the proof.

The mapping $(\cdot, \cdot|_{\bar{p}}) : R^n \rightarrow R$ given by

$$(\bar{x}, \bar{y}|\bar{z})_{\bar{p}} = \frac{1}{2} \sum_{i,j=1}^n p_i p_j (x_i z_j - x_j z_i)(y_i z_j - y_j z_i),$$

where $\bar{x} = (x_1, x_2, \dots, x_n)$, $\bar{y} = (y_1, y_2, \dots, y_n)$, $\bar{z} = (z_1, z_2, \dots, z_n) \in R^n$ and $\bar{p} = (p_1, p_2, \dots, p_n) > \bar{0}$, that is, $p_i > 0$ for all $i = 1, 2, \dots, n$, is obviously a 2-inner product on R^n generating the 2-norm on R^n

$$\|\bar{x}, \bar{y}\|_{\bar{p}} = \left[\frac{1}{2} \sum_{i,j=1}^n p_i p_j (x_i z_j - x_j z_i)^2 \right]^{1/2}.$$

Proposition 2.2. *Let $(R^n, (\cdot, \cdot|_{\bar{p}}))$ be a 2-inner product space and $\bar{x}, \bar{y}, \bar{z}, \bar{e} \in R^n$ such that $\|\bar{e}, \bar{z}\| = 1$ and $\bar{z} \notin V(\bar{x}, \bar{y}, \bar{e})$. If m, n, M, N are real numbers such that*

$$(M\bar{e} - \bar{x}, \bar{x} - m\bar{e}|\bar{z})_{\bar{p}} \geq 0, \quad (N\bar{e} - \bar{y}, \bar{y} - n\bar{e}|\bar{z})_{\bar{p}} \geq 0,$$

then we have the inequality

$$\begin{aligned} & \left| \frac{1}{2} \sum_{i,j=1}^n p_i p_j (x_i z_j - x_j z_i)(y_i z_j - y_j z_i) \right. \\ & \quad - \left(\frac{1}{2} \sum_{i,j=1}^n p_i p_j (x_i z_j - x_j z_i)(e_i z_j - e_j z_i) \right) \\ & \quad \times \left. \left(\frac{1}{2} \sum_{i,j=1}^n p_i p_j (e_i z_j - e_j z_i)(y_i z_j - y_j z_i) \right) \right| \\ & \leq \frac{1}{4}|M - m||N - n|. \end{aligned}$$

Next, let (\cdot, \cdot) be a 2-inner product and $\{(\cdot, \cdot)_i\}_{i \in N}$ be a sequence of 2-inner products satisfying the following condition:

$$\|x, z\|^2 > \sum_{i=1}^{\infty} \|x, z\|_i^2$$

for all x, z being linearly independent. Let $p \in N$. Define a mapping

$$(x, y|z)_p = (x, y|z) - \sum_{i=1}^p (x, y|z)_i,$$

for $x, y, z \in X$ and $z \notin V(x, y)$. Then the mapping $(\cdot, \cdot)_p$ satisfies the properties:

- (1) $(x, x|z)_p \geq 0$,
- (2) $(\alpha x + \beta x', y|z)_p = \alpha(x, y|z)_p + \beta(x' + y|z)_p$,
- (3) $(x, y|z)_p + (y, x|z)_p$,
- (4) $(x, x|z)_p = (z, z|x)_p$

for every $x, x', y, z \in X$ and $\alpha, \beta \in R$.

By Theorem 2.1, we have the following:

Proposition 2.3. *If there exist real numbers m, n, M, N are real numbers such that*

$$(Me - x, x - me|z)_p \geq 0, \quad (Ne - y, y - ne|z)_p \geq 0,$$

then we have

$$|(x, y|z)_p - (x, e|z)_p(e, y|z)_p| \leq \frac{1}{4}|M - m||N - n|.$$

3. Applications for Isotonic functionals

Let E be a nonempty set, $F(E, R)$ be the real algebra of all real-valued functions defined on E and L be a subalgebra of $F(E, R)$. A functional A

is said to be *isotonic* if $f \geq g$, that is, $f(t) \geq g(t)$ for every $t \in E$, implies $A(f) \geq A(g)$ for all $f, g \in L$. A functional A is said to be *normalized* on L if $\mathbf{1} \in L$, that is, $\mathbf{1}(t) = 1$ for all $t \in E$ implies $A(\mathbf{1}) = 1$.

For some inequalities involving linear isotonic functionals is given in [8].

Suppose that $fgh^2, fh^2, gh^2 \in L$ for all $f, g \in L$. For a isotonic linear functional $A : L \rightarrow R$, we define a functional $(\cdot, \cdot | \cdot)_A : L \times L \times L \rightarrow R$ by

$$(f, g | h)_A = A(fgh^2)$$

for every $f, g, h \in L$. Then we have the following properties:

- (1) $(f, f | h)_A = A(f^2h^2) \geq 0$,
- (2) $(\alpha f + \beta f', g | h)_A = \alpha(f, g | h)_A + \beta(f', g | h)_A$,
- (3) $(f, g | h)_A = (g, f | h)_A$,
- (4) $(f, f | h)_A = (h, h | f)_A$.

for every $f, f', g, h \in L$ and $\alpha, \beta \in R$.

Theorem 3.1. *Let L be as above, $fgh^2, fh^2, gh^2, f, g, e, h \in L$ with $\|e, h\| = 1$ and $h \notin V(f, g, e)$. If m, n, M, N are real numbers such that*

$$(3.1) \quad m \leq f \leq M, \quad n \leq g \leq N$$

and $A : L \rightarrow R$ is an isotonic linear functional, then we have the following inequality

$$|A(fgh^2) - A(fh^2)A(gh^2)| \leq \frac{1}{4}(M - m)(N - n).$$

Proof. Choose $e = \mathbf{1}$. Then since $\|e, h\| = 1$, $(e, e | h)_A = 1$, $A(e^2h^2) = A(h^2) = 1$ and we have

$$(Me - f, f - me | h)_A = A((M - f)(f - m)h^2) \geq 0$$

and

$$(Ne - g, g - ne | h)_A = A((N - g)(g - n)h^2) \geq 0.$$

Applying Theorem 2.1 for $(\cdot, \cdot)_A$, we have

$$|(f, g|h)_A - (f, e|h)(e, g|h)_A| \leq \frac{1}{4}(M - m)(N - n).$$

This completes the proof.

Corollary 3.2. *Let $f, g, e, h \in L$. Suppose $\mathbf{1} \in L$ and $A : L \rightarrow R$ is a normalized isotonic linear functional. If m, n, M, N satisfy (3.1), then we have the following inequality*

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4}(M - m)(N - n).$$

Let $L_{[a,b]}^2$ be a real Hilbert space of square integrable mapping on $[a, b]$, that is, $\int_a^b |f^2| dm < \infty$ if $f \in L_{[a,b]}^2$. Define a mapping $(\cdot, \cdot) : L_{[a,b]}^2 \times L_{[a,b]}^2 \times L_{[a,b]}^2 \rightarrow R$ by

$$\begin{aligned} & (f, g|l) \\ &= \frac{1}{2} \int_a^b \int_a^b (f(x)l(y) - f(y)l(x))(g(x)l(y) - g(y)l(x)) dm(x)dm(y). \end{aligned}$$

Then (\cdot, \cdot) is a 2-inner product on $L_{[a,b]}^2$ generating the 2-norm

$$\|f, l\| = \left(\frac{1}{2} \int_a^b \int_a^b (f(x)l(y) - f(y)l(x))^2 dm(x)dm(y) \right)^{1/2}.$$

Proposition 3.3. *Let $(L_{[a,b]}^2, (\cdot, \cdot))$ be a 2-inner product space and $f, g, e, h \in L_{[a,b]}^2$ with $\|e, h\| = 1$ and $h \notin V(f, g, e)$. There exist real numbers m, n, M, N such that*

$$m \leq f \leq M, \quad n \leq g \leq N.$$

Then we have the inequality

$$\begin{aligned} & \left| \int_a^b \int_a^b (f(x)h(y) - f(y)h(x))(g(x)h(y) - g(y)h(x))dm(x)dm(y) \right. \\ & \quad \left. - \left[\int_a^b \int_a^b (f(x)h(y) - f(y)h(x))(h(y) - h(x))dm(x)dm(y) \right] \right. \\ & \quad \left. \times \left[\int_a^b \int_a^b (h(y) - h(x))(g(x)h(y) - g(y)h(x))dm(x)dm(y) \right] \right| \\ & \leq \frac{1}{4}|M - m||N - n|. \end{aligned}$$

Proof. By Theorem 3.1, applied to

$$A(f, g|h) = A(fgh^2) = \int_a^b \int_a^b \det(f, h) \det(g, h)dm(x)dm(y),$$

where

$$\det(f, h) = \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix},$$

the result follows.

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