A CONVOLUTED FIBONACCI SEQUENCE - PART II

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ABSTRACT. We consider a generalisation of the classical Fibonacci sequence, and by the use of function theoretic methods, generate binomial type series which may be expressed in closed form. Some new identities are also given.

1. INTRODUCTION

In this paper we will extend the ideas developed in our previous paper [2]. We consider an arbitrary order difference scheme and by the use of Z transform theory generate binomial type sums that may be represented in closed form. In particular, we consider multiple zeros of an associated polynomial characteristic function and following the methods of our previous paper [2], we shall generalise a result given by Wilf [3]. We also employ Zeilberger's creative telescoping algorithm, Petkovšek's algorithm 'Hyper' and Wilf and Zeilberger's WZ pairs method to certify particular instances of the generated binomial sums. Finally, we generalise our results by considering forcing terms of binomial type.

2. TECHNIQUE

For the sake of completeness, we shall describe the technique as given in [2]. Consider a generalised Fibonacci sequence f_n, that satisfies

$$\left.\begin{array}{r} \sum\limits_{j=0}^{R} \binom{R}{R-j} (-c)^{R-j} \sum\limits_{r=0}^{j} \binom{j}{r} (-b)^{j-r} f_{n+r-(R-j)a} = w_n;\ n \geq aR \\ \sum\limits_{r=0}^{R} \binom{R}{r} (-b)^{R-r} f_{n+r} = w_n;\ \ n < aR \end{array}\right\} \tag{2.1}$$

with a and R integer, b and c real and w_n is a discrete forcing term. A method of analyzing the solution of system (2.1) is by the use of Z transform techniques. Without loss of generality, let $w_n = 0,\ f_{R-1} = 1$ and all other initial conditions of the system (2.1) be zero. If we now take the Z transform of (2.1), utilize the two Z transform properties

$$Z\left[f_{n+k}\right] = z^k \left[f(z) - \sum_{n=0}^{k-1} f_n z^{-n} \right]$$

and

$$Z\left[f_{n-k} U_{n-k}\right] = z^{-k} F(z),$$

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where U_{n-k} is the discrete step function, we obtain

$$F(z)\left\{\sum_{j=0}^{R}\binom{R}{j}(z-b)^{j}\left(-cz^{-a}\right)^{R-j}\right\}=z. \tag{2.2}$$

From (2.2)

$$F(z)=\frac{z}{\left(z-b-cz^{-a}\right)^{R}}=\frac{z^{aR+1}}{\left(z^{a+1}-bz^{a}-c\right)^{R}}. \tag{2.3}$$

In series form, (2.3) may be expressed as

$$F(z)=\sum_{r=0}^{\infty}\binom{R+r-1}{r}\frac{c^{r}z^{1-ar}}{(z-b)^{R+r}} \tag{2.4}$$

and we may obtain the inverse Z transform of (2.4) such that

$$\begin{aligned} f_n(a,b,c,R) &= f_n \\ &= \sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}\left(\frac{c}{b}\right)^{r}b^{n-ar-R+1} \end{aligned} \tag{2.5}$$

where $[x]$ represents the integer part of x. The inverse Z transform of (2.3) may also be expressed as

$$f_n=\frac{1}{2\pi i}\oint_C z^{n}\left(\frac{F(z)}{z}\right)dz=\sum_{j=0}^{a}z^{n}\mathrm{Res}_j\left(\frac{F(z)}{z}\right), \tag{2.6}$$

where C is a smooth Jordan curve enclosing the singularities of (2.3) and Res_j is the residue of the poles of (2.3). The residue, Res_j, of (2.6) depend on the zeros of the characteristic function in (2.3), namely

$$g(z)=z^{a+1}-bz^{a}-c. \tag{2.7}$$

Now, $g(z)$ has $a+1$ distinct zeros ξ_j, $j=0,1,2,3,...,a$, for

$$c\neq -a^{a}\left(\frac{b}{a+1}\right)^{a+1}$$

therefore the singularities in (2.3) are all poles of order R. We may now write (2.6) as

$$f_n=\sum_{j=0}^{a}\sum_{\mu=0}^{R-1}Q_{R,\mu}\left(\xi_j\right)\binom{n}{R-1-\mu}\xi_j^{n-R+1+\mu} \tag{2.8}$$

where

$$\mu!Q_{R,\mu}\left(\xi_j\right)=\lim_{z\to\xi_j}\left[\frac{d^{\mu}}{dz^{\mu}}\left\{\left(z-\xi_j\right)^{R}\frac{F(z)}{z}\right\}\right] \tag{2.9}$$

for each $j=0,1,2,3,...,a$, and $F(z)$ is given by (2.3). Combining the expressions in (2.5) and (2.8) we have that

$$\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}\left(\frac{c}{b}\right)^{r}b^{n-ar-R+1}$$

$$= \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n}{R-1-\mu} \xi_j^{n-R+1+\mu} \tag{2.10}$$

and putting $n = n^*(a+1) + R - 1$ in (2.10) and renaming n^* as n, we have an alternate form

$$\begin{aligned} &\sum_{r=0}^{n} \binom{R+r-1}{r} \binom{n(a+1)+R-1-ar}{R+r-1} \left(\frac{c}{b}\right)^r b^{n(a+1)-ar} \\ = &\sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n(a+1)+R-1}{R-1-\mu} \xi_j^{n(a+1)+\mu}. \end{aligned} \tag{2.11}$$

Let us now consider the case of multiple zeros of the characteristic function (2.7). In doing so, we shall recover a result given by Wilf [3], and describe a generalisation of this result which we believe to be new. The WZ pairs method of Wilf and Zeilberger will also be employed to certify particular instances of identities that we shall generate.

3. Multiple Zeros

When the characteristic function (2.7) has double (repeated) zeros, which will be the case for $c = -a^a \left(\frac{b}{a+1}\right)^{a+1}$, then (2.3) has poles of order $2R$. In this case we may write, from (2.11)

$$\begin{aligned} &\sum_{r=0}^{n} \binom{R+r-1}{r} \binom{n(a+1)+R-1-ar}{R+r-1} \left(\frac{-a^a}{(a+1)^{a+1}}\right)^r \\ = \; & b^{-n(a+1)} \sum_{j=0}^{a} z^n \mathrm{Res}_j \left(\frac{F(z)}{z}\right) \end{aligned} \tag{3.1}$$

where the Res_j must take into account the repeated zeros of (2.7).

For $a = 1, c = -(b/2)^2$ and, from (2.3),

$$F(z) = \frac{z^{R+1}}{(z-b/2)^{2R}}$$

which has poles of order $2R$ at $z = b/2$. Utilizing (2.8), (2.9) and (3.1) we have

$$\begin{aligned} f_n(R) &= \sum_{r=0}^{n} \binom{R+r-1}{r} \binom{2n+R-1-r}{R+r-1} \left(\frac{-1}{4}\right)^r \\ &= 2^{-2n} \sum_{\mu=0}^{R} \binom{R}{\mu} \binom{2n+R-1}{2R-1-\mu}. \end{aligned} \tag{3.2}$$

If $R = 1$, then (3.2) reduces to a result given on page 124 of Wilf's book [3], namely

$$\sum_{r=0}^{n} \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r = 2^{-2n}(2n+1) = \prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{2n+1}\right). \tag{3.3}$$

Hence (3.2) is a generalisation of (3.3) which we believe to be new. Utilizing Zeilberger's creative telescoping algorithm, described in [2] and available on 'Mathematica', we obtain from the left hand side of (3.2) a recurrence $f_n(R)$ that satisfies

$$4(n+1)(2n+1) f_{n+1}(R) - (n+R)(2n+2R+1) f_n(R) = 0. \tag{3.4}$$

Iterating (3.4), we have that

$$f_n(R) = 2^{-2n} \prod_{j=0}^{n-1} \frac{(R+j)(2R+1+2j)}{(1+j)(1+2j)} \tag{3.5}$$

so that from (3.2) and (3.5) we obtain

$$\begin{aligned} &2^{2n} \sum_{r=0}^{n} \binom{R+r-1}{r} \binom{2n+R-1-r}{R+r-1} \left(\frac{-1}{4}\right)^r \\ =\ & \sum_{\mu=0}^{R} \binom{R}{\mu} \binom{2n+R-1}{2R-1-\mu} = \prod_{j=0}^{n-1} \frac{(R+j)(2R+1+2j)}{(1+j)(1+2j)}. \end{aligned} \tag{3.6}$$

Further results may be obtained as follows. Differentiate (2.11), for $R = 1$, and its trigonometric representation with respect to c, then substitute $c = -(b/2)^2$ and simplify such that

$$\begin{aligned} f_n'(1) &= \sum_{r=1}^{n} r \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r \\ &= -\prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{2n+1}\right) \sum_{k=1}^{n} \cot^2\left(\frac{\pi k}{2n+1}\right). \end{aligned} \tag{3.7}$$

From 'Mathematica', a recurrence relation for $f_n'(1)$ in (3.7) is

$$4n(2n-1) f_{n+1}'(1) - (n+1)(2n+3) f_n'(1) = 0. \tag{3.8}$$

Iterating (3.8) and using (3.7) we have

$$\sum_{r=1}^{n} r \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r = -2^{-2n} \prod_{j=1}^{n-1} \frac{(1+j)(3+2j)}{j(2j-1)} \tag{3.9}$$

and comparing (3.7) and (3.9), we have

$$2^{-2n} \prod_{j=1}^{n-1} \frac{(1+j)(3+2j)}{j(2j-1)} = \prod_{j=1}^{n} \sin^2\left(\frac{\pi j}{2n+1}\right) \sum_{k=1}^{n} \cot^2\left(\frac{\pi k}{2n+1}\right). \tag{3.10}$$

To further illustrate the technique, from (3.2) and (3.4) with $R = 2$ we obtain

$$\begin{aligned} &\sum_{r=0}^{n} \binom{r+1}{r} \binom{2n+1-r}{r+1} \left(\frac{-1}{4}\right)^r \\ =\ & 2^{-2n} \left\{ \binom{2n+1}{3} + 2\binom{2n+1}{2} + \binom{2n+1}{1} \right\} \\ =\ & 2^{-2n} \prod_{j=0}^{n-1} \frac{(2+j)(5+2j)}{(1+j)(2j+1)}. \end{aligned} \tag{3.11}$$

Writing

$$\sum_{r=0}^{n} \binom{r+1}{r} \binom{2n+1-r}{r+1} \left(\frac{-1}{4}\right)^r = \sum_{r=0}^{n} (2n+1-r) \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r$$

and using result (3.2) we have that

$$\begin{aligned}(3.12)\qquad & \sum_{r=0}^{n}\binom{r+1}{r}\binom{2n+1-r}{r+1}\left(\frac{-1}{4}\right)^{r} \\ = {} & 2^{-2n}\left(2n+1\right)^{2}+\prod_{j=1}^{n}\sin^{2}\left(\frac{\pi j}{2n+1}\right)\sum_{k=1}^{n}\cot^{2}\left(\frac{\pi k}{2n+1}\right).\end{aligned}$$

From (3.11) and (3.12) the identity

$$\begin{aligned}(3.13)\qquad & 2^{-2n}\prod_{j=0}^{n-1}\frac{\left(2+j\right)\left(5+2j\right)}{\left(1+j\right)\left(2j+1\right)}-\prod_{j=1}^{n}\sin^{2}\left(\frac{\pi j}{2n+1}\right)\sum_{k=1}^{n}\cot^{2}\left(\frac{\pi k}{2n+1}\right) \\ = {} & 2^{-2n}\left(2n+1\right)^{2}\end{aligned}$$

is obtained and rewriting we have, using (3.10), that

$$\left(2n+1\right)^{2}=10\prod_{j=1}^{n-1}\frac{\left(2+j\right)\left(5+2j\right)}{\left(1+j\right)\left(2j+1\right)}-\prod_{j=1}^{n-1}\frac{\left(1+j\right)\left(3+2j\right)}{j\left(2j-1\right)}.$$

From (3.10) and (3.11)

$$\frac{n\left(4n^{2}-1\right)}{3}=\prod_{j=1}^{n-1}\frac{\left(1+j\right)\left(3+2j\right)}{j\left(2j-1\right)},$$

$$\frac{\left(n+1\right)\left(2n+1\right)\left(2n+3\right)}{30}=\prod_{j=1}^{n-1}\frac{\left(2+j\right)\left(5+2j\right)}{\left(1+j\right)\left(2j+1\right)}$$

and from (3.7)

$$\sum_{r=1}^{n}r\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r}=-\frac{2^{-2n}n\left(4n^{2}-1\right)}{3}.$$

Similarly, we can show that

$$\begin{aligned}(3.14)\qquad f_{n} &= \sum_{r=1}^{n}r^{2}\binom{2n-r}{r}\left(\frac{-1}{4}\right)^{r} \\ &= \frac{2^{-2n}n\left(8n^{4}-20n^{3}-10n^{2}+5n+2\right)}{15} \\ &= -2^{-2n}\prod_{j=1}^{n-1}\frac{\left(j+1\right)\left(2j+3\right)\left(2j^{2}-j-5\right)}{j\left(2j-1\right)\left(2j^{2}-5j-2\right)}.\end{aligned}$$

The left hand side of (3.14) satisfies the recurrence

$$4n\left(2n-1\right)\left(2n^{2}-5n-2\right)f_{n+1}+\left(n+1\right)\left(2n+3\right)\left(2n^{2}-n-5\right)f_{n}=0$$

and hence

$$\frac{n\left(8n^{4}-20n^{3}-10n^{2}+5n+2\right)}{15}=-\prod_{j=1}^{n-1}\frac{\left(j+1\right)\left(2j+3\right)\left(2j^{2}-j-5\right)}{j\left(2j-1\right)\left(2j^{2}-5j-2\right)}.$$

Similarly

$$\begin{aligned}
&\sum_{r=1}^{n} r^3 \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r \\
= &\ \frac{2^{-2n} n \left(16n^6 - 112n^5 + 112n^4 + 140n^3 - 21n^2 - 28n - 2\right)}{105} \\
= &\ -2^{-2n} \prod_{j=1}^{n-1} \frac{(j+1)(2j+3)\left(4j^4 - 12j^3 - 31j^2 + 18j + 35\right)}{j(2j-1)(4j^4 - 28j^3 + 29j^2 + 28j + 2)}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
&\prod_{j=1}^{n-1} \frac{(j+1)(2j+3)\left(4j^4 - 12j^3 - 31j^2 + 18j + 35\right)}{j(2j-1)(4j^4 - 28j^3 + 29j^2 + 28j + 2)} \\
= &\ \frac{n\left(16n^6 - 112n^5 + 112n^4 + 140n^3 - 21n^2 - 28n - 2\right)}{105}.
\end{aligned}$$

In general

$$\frac{2^{2n}}{n} \sum_{r=1}^{n} r^m \binom{2n-r}{r} \left(\frac{-1}{4}\right)^r$$

can be expressed as a polynomial in n of degree $2m$ for m integer. By the WZ package on 'Mathematica' the identity (3.14) may be verified by the certificate function

$$V(n,r) = \frac{2(r-1)(r-1-2n)\left(\begin{array}{c} 12n^4(r-1) - 8n^3\left(r^2 - r + 3\right) + n^2\left(4r^2 - 15r - 13\right) \\ +2rn(6r-7) + 6r^2 - 5r + 1 \end{array}\right)}{r(2r-1-2n)(r-1-n)(n+1)(2n+3)(2n^2-n-5)}.$$

Similarly for the identity (3.2), for particular values of R, and by the use of the WZ package we may obtain a rational certificate function, $V(n,r,R)$ that certifies the identity, in particular

$$V(n,r,1) = \frac{2r(2n+1-r)(4r-5-6n)}{(2n+3)(2n+1-2r)(n+1-r)} \text{ and}$$

$$V(n,r,4) = \frac{2r(2n+4-r)\left(4nr + 10r - 6n^2 - 23n - 14\right)}{(n+4)(n+9)(2n+1-2r)(n+1-r)}.$$

More Sums.

Since (2.7) has at most three real zeros we may obtain further results as follows. Consider multiple zeros of (2.7) for $a = 2$ and $c = -4(b/3)^3$ such that $g(z) = \left(z - \frac{2b}{3}\right)^2 \left(z + \frac{b}{3}\right)$ and therefore (2.3) and (2.9) may be modified such that

$$F(z) = \frac{z^{2R+1}}{\left(\left(z - \frac{2b}{3}\right)^2 \left(z + \frac{b}{3}\right)\right)^R},$$

$$\mu! Q_{2R,\mu}\left(\frac{2b}{3}\right) = \lim_{z \to \frac{2b}{3}} \left[\frac{d^\mu}{dz^\mu}\left\{\left(z - \frac{2b}{3}\right)^{2R} \frac{F(z)}{z}\right\}\right], \mu = 0, 1, 2, ..., 2R-1,$$

$$\nu! P_{R,\nu}\left(-\frac{b}{3}\right) = \lim_{z\to -\frac{b}{3}}\left[\frac{d^{\nu}}{dz^{\nu}}\left\{\left(z+\frac{b}{3}\right)^{R}\frac{F(z)}{z}\right\}\right], \nu = 0,1,2,...,R-1$$

and hence from (2.11)

$$\sum_{r=0}^{n}\binom{R+r-1}{r}\binom{3n+R-1-2r}{R+r-1}\left(\frac{-4}{27}\right)^{r} b^{3n} \tag{3.15}$$
$$= \sum_{j=0}^{a}\sum_{\mu=0}^{2R-1} Q_{2R,\mu}\left(\frac{2b}{3}\right)\binom{3n+R-1}{2R-1-\mu}\left(\frac{2b}{3}\right)^{3n+\mu}$$
$$+\sum_{j=0}^{a}\sum_{\nu=0}^{R-1} P_{R,\nu}\left(-\frac{b}{3}\right)\binom{3n+R-1}{R-1-\mu}\left(-\frac{b}{3}\right)^{3n+\mu}.$$

For $R=1$ and $R=2$, we have respectively from (3.15) that

$$f_n(1) = \sum_{r=0}^{n}\binom{3n-2r}{r}\left(\frac{-4}{27}\right)^{r} \tag{3.16}$$
$$= 3^{-(3n+2)}\left\{2^{3n+1}(9n+4)+(-1)^{n}\right\} \text{ and}$$

$$f_n(2) = \sum_{r=0}^{n}(r+1)\binom{3n+1-2r}{r+1}\left(\frac{-4}{27}\right)^{r} \tag{3.17}$$
$$= 3^{-(3n+2)}\left\{\begin{array}{c} 2^{3n+2}\binom{3n+1}{3} + \frac{2^{3n+4}}{27} + \\ \frac{2^{3n+5}}{3}\binom{3n+1}{2} + 2^{3n+3}(3n+1) \\ +\frac{(-1)^{n}}{9}\left(3n+\frac{11}{3}\right)\end{array}\right\}$$
$$= (3n+1)\ {}_3F_2\left[\left.\begin{array}{c}\frac{1-3n}{3}, \frac{2-3n}{3}, -n \\ \frac{-1-3n}{2}, \frac{-3n}{2}\end{array}\right| 1\right].$$

From 'Hyper', in 'Mathematica' a recurrence relation for (3.16) and (3.17) is, respectively

$$729(3n+4) f_{n+2}(1) - 27(21n+52) f_{n+1}(1) - 8(3n+7) f_n(1) = 0,$$
$$f_0(1) = 1,\ f_1(1) = \frac{23}{27}$$

and

$$729(3n+5)(3n+4)^{2} f_{n+2}(2) - 27\left(189n^{3}+1440n^{2}+3399n+2348\right) f_{n+1}(2)$$
$$-8(3n+7)(3n+8)(3n+10) f_n(2) = 0,$$
$$f_0(2) = 1, f_1(2) = \frac{100}{27}.$$

4. Other forcing terms.

We can now consider the system (2.1) with non zero forcing terms. Consider a forcing term of the form, (other forms may also be taken).

$$w_n = \binom{n}{m+R-1} b^{n+1-R-m}$$

with all initial conditions zero and m a positive integer, again the results of the previous section are applicable. For the purpose of demonstration let $a=1$ and $c=-(b/2)^2$ so that from (3.1)

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{R+r-1}{r}\binom{2n+R+m-1-r}{R+m+r-1}\left(\frac{-1}{4}\right)^r\\
=\;& b^{-2n}\sum_{\mu=0}^{2R-1}Q_{2R,\mu}\left(\frac{b}{2}\right)\binom{n}{2R-1-\mu}\left(\frac{b}{2}\right)^{2n-R+m+\mu}\\
&+\sum_{\nu=0}^{m-1}P_{m,\nu}(b)\binom{n}{m-1-\mu}b^{R+\mu}
\end{aligned}$$

where

$$\mu!Q_{2R,\mu}\left(\frac{b}{2}\right)=\lim_{z\to\frac{b}{2}}\left[\frac{d^\mu}{dz^\mu}\left\{\frac{z^R}{(z-b)^R}\right\}\right]$$

and

$$\nu!P_{m,\nu}(b)=\lim_{z\to b}\left[\frac{d^\nu}{dz^\nu}\left\{\frac{z^R}{\left(z-\frac{b}{2}\right)^{2R}}\right\}\right].$$

In the case that $R=1,m=1$ and 2 respectively we obtain

$$\sum_{r=0}^{n}\binom{2n+1-r}{r+1}\left(\frac{-1}{4}\right)^r=4-2^{-2n}(2n+3)=4-3.2^{-2n}\prod_{j=0}^{n-1}\frac{5+2j}{3+2j}$$

and

$$\begin{aligned}
\sum_{r=0}^{n}\binom{2n+2-r}{r+2}\left(\frac{-1}{4}\right)^r &= 4(2n-1)+2^{-2n}(2n+5)\\
&= 4\prod_{j=1}^{n-1}\frac{2j+1}{2j-1}+7.2^{-2n}\prod_{j=1}^{n-1}\frac{7+2j}{5+2j}\\
&= (n+1)(2n+1)_3F_2\left[\begin{array}{c}1,\frac{1-2n}{2},-n\\3,-2-2n\end{array}\middle|1\right]
\end{aligned}$$

For constants α_j and positive integer m we have that

$$f_n=\sum_{r=0}^{n}\binom{2n+m-r}{r+m}\left(\frac{-1}{4}\right)^r=(-1)^m2^{-2n}(2n+2m+1)+4\sum_{j=0}^{n-1}\alpha_jn^j$$

and for $m=0$ reduces to identity (3.3); moreover a recurrence for the left hand side is

$$\begin{aligned}
&4(2n+2m+1)f_{n+1}-(2n+2m+3)f_n\\
=\;&\frac{4m(6n+2m+5)}{(2n+m+2)}\binom{2n+m+2}{m},\quad f_0=1.
\end{aligned}$$

5. Conclusion.

We have shown that many identities of binomial type sums may be generated by an application of the Z transform.

References

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