BOUNDS FOR SOLUTIONS OF CERTAIN FINITE DIFFERENCE EQUATIONS

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Abstract. In the present paper we obtain bounds on the solutions of certain finite difference equations. The discrete analogue of Bihari's inequality is used to obtain the results.

1. Introduction

In this paper we first consider the following second order nonlinear difference equation

$$\Delta^2 u(n) + f(n, u(n), \Delta u(n)) = 0, \tag{1.1}$$

with the given initial conditions

$$u(1) = c_1, \quad \Delta u(1) = c_2, \tag{1.2}$$

for $n \in N$, where $N = \{1, 2, \dots\}$, Δ is the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n)$, c_1, c_2, are constants and $f : N \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, in which $\mathbb{R} = (-\infty, \infty)$.

In the past few years many results concerning the properties of the solutions of (1.1) and its various special versions have been established by using different techniques, see [3], [6], [7], [8] and [10] and the references cited therein. Our main objective in this paper is to obtain bounds on the solutions of (1.1) – (1.2) by using some suitable conditions on the function f involved in (1.1). The discrete analogue of the Bihari's inequality is used to establish our results. Further extensions of the main results to third order difference equations and the equations of the more general type are also given. In fact, our results are motivated by the recent work of A. Constantin [1] concerning the asymptotic behaviour of solutions of second order nonlinear differential equations and some of the references cited therein.

2. Statement of Results

The following result established in 1960 by T.E. Hull and W.A.J. Luxemburg [2, p. 36], which in turn is a discrete version of Bihari's inequality is useful in the proofs of the theorems to follow.

Lemma 1. *Let $x(n)$ and $p(n)$ be real-valued nonnegative functions defined for $n = 0, 1, 2, \dots,$ and let g be continuous positive and nondecreasing on an interval*

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$I = [u_0, \infty)$, $u_0 > 0$ is a constant, and suppose further that the inequality

$$x(n) \leq k + \sum_{s=0}^{n-1} p(s)\, g(x(s)),$$

is satisfied for all $n = 0, 1, 2, \ldots,$ where k is a nonnegative constant. Then

$$x(n) \leq G^{-1}\left[G(k) + \sum_{s=0}^{n-1} p(s)\right], \quad 0 \leq n \leq \alpha,$$

where

$$G(u) = \int_{u_0}^{u} \frac{ds}{g(s)}, \quad u \geq u_0 > 0,$$

G^{-1} is the inverse of G, and

$$\alpha = \sup\left\{n : G(k) + \sum_{s=0}^{n-1} p(s) \in Dom\left(G^{-1}\right)\right\}.$$

For the proof of this lemma we refer the interested readers to Hull and Luxemberg [2], see also [4] and [5].

Our main result reads as follows.

Theorem 1. *Suppose that there exist functions $h_i : N \to \mathbb{R}_+$, $(i = 1, 2, 3)$ such that*

$$|f(n, u, v)| \leq h_1(n)\, g\left(\frac{|u|}{n}\right) + h_2(n)\,|v| + h_3(n), \tag{2.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function such that $g(u) > 0$ for $u > 0$ and define

$$G(r) = \int_{1}^{r} \frac{ds}{g(s)}, \quad r > 0. \tag{2.2}$$

Then for every solution $u(n)$ of (1.1) – (1.2) we have

$$\begin{aligned} |u(n)| \leq\ & nG^{-1}\left[G\left(a(n)\prod_{s=1}^{n-1}[1 + h_2(s)]\right)\right. \\ & \left. + \left(\prod_{s=1}^{n-1}[1 + h_2(s)]\right)\sum_{s=1}^{n-1} h_1(s)\left(\prod_{\sigma=1}^{s}[1 + h_2(\sigma)]^{-1}\right)\right], \end{aligned} \tag{2.3}$$

for $1 \leq n \leq \alpha_1$, where G^{-1} is the inverse of G,

$$a(n) = 1 + |c_1| + |c_2| + \sum_{s=1}^{n-1} h_3(s), \tag{2.4}$$

and

$$\begin{aligned} \alpha_1 = \sup & \left\{n \in N : G\left(a(n)\prod_{s=1}^{n-1}[1 + h_2(s)]\right) + \left(\prod_{s=1}^{n-1}[1 + h_2(s)]\right)\right. \\ & \left. \times \sum_{s=1}^{n-1} h_1(s)\left(\prod_{\sigma=1}^{s}[1 + h_2(\sigma)]^{-1}\right) \in Dom\left(G^{-1}\right)\right\}. \end{aligned}$$

A slight variant of Theorem 1 is given in the following theorem.

Theorem 2. *Suppose that there exist functions $h_i : N \to \mathbb{R}_+$, $(i = 1, 2, 3)$ such that*

$$|f(n, u, v)| \le h_1(n) \frac{|u|}{n} + +h_2(n) g(|v|) + h_3(n), \tag{2.5}$$

where g is as defined in Theorem 1. Let G and G^{-1} also be as defined in Theorem 1. Then for every solution $u(n)$ of (1.1) – (1.2) we have

$$\begin{aligned} |u(n)| \quad &\le \quad nG^{-1}\left[G\left(a(n)\prod_{s=1}^{n-1}[1 + h_1(s)]\right)\right. \\ &\left. + \left(\prod_{s=1}^{n-1}[1 + h_1(s)]\right)\sum_{s=1}^{n-1} h_2(s)\left(\prod_{\sigma=1}^{s}[1 + h_1(\sigma)]\right)\right], \end{aligned} \tag{2.6}$$

for $1 \le n \le \alpha_2$, where $a(n)$ is defined by (2.4) in Theorem 1 and

$$\begin{aligned} \alpha_2 \quad &= \quad \sup\left\{n \in N : G\left(a(n)\prod_{s=1}^{n-1}[1 + h_1(s)]\right) + \left(\prod_{s=1}^{n-1}[1 + h_1(s)]\right)\right. \\ &\left. \times \sum_{s=1}^{n-1} h_2(s)\left(\prod_{\sigma=1}^{s}[1 + h_1(\sigma)]\right) \in Dom\left(G^{-1}\right)\right\}. \end{aligned}$$

Remark 1. *We note that our results in Theorems 1 and 2 gives in the special cases the bounds on the solutions of the equations of the form*

$$\Delta^2 u(n) + f(n, u(n)) = 0, \tag{2.7}$$

under the given initial conditions (1.2). For a number of related recent results concerning the equations of the above forms (1.1) and (2.7), see [3] *and* [7].

3. Proofs of Theorems 1 and 2

Proof. From (1.1) – (1.2) it is easy to observe that

$$\Delta u(n) = c_2 - \sum_{s=1}^{n-1} f(s, u(s), \Delta u(s)), \quad n \in N, \tag{3.1}$$

and

$$u(n) = c_1 + (n - 1)c_2 - \sum_{s=1}^{n-1}(n - s) f(s, u(s), \Delta u(s)), \quad n \in N. \tag{3.2}$$

Using (2.1) in (3.1) and (3.2) we have

$$|\Delta u(n)| \le |c_2| + \sum_{s=1}^{n-1}\left[h_1(s) g\left(\frac{|u(s)|}{s}\right) + h_2(s)|\Delta u(s)| + h_3(s)\right], \tag{3.3}$$

and

$$\frac{|u(n)|}{n} \le |c_1| + |c_2| + \sum_{s=1}^{n-1}\left[h_1(s) g\left(\frac{|u(s)|}{s}\right) + h_2(s)|\Delta u(s)| + h_3(s)\right]. \tag{3.4}$$

Define a function $x(n)$ by

$$x(n) = |c_1| + |c_2| + \sum_{s=1}^{n-1}\left[h_1(s) g\left(\frac{|u(s)|}{s}\right) + h_2(s)|\Delta u(s)| + h_3(s)\right]. \tag{3.5}$$

Now, using the facts that $|\Delta u(n)| \leq x(n)$ and $\frac{|u(n)|}{n} \leq x(n)$ in (3.5) we observe that

$$x(n) \leq a(n) + \sum_{s=1}^{n-1} [h_1(s) g(x(s)) + h_2(s) x(s)], \tag{3.6}$$

where $a(n)$ is defined by (2.4).

For a fixed element $m \in N$, from (3.6) we observe that

$$x(n) \leq a(m) + \sum_{s=1}^{n-1} [h_1(s) g(x(s)) + h_2(s) x(s)], \tag{3.7}$$

for $1 \leq n \leq m$. Put

$$y(n) = a(m) + \sum_{s=1}^{n-1} [h_1(s) g(x(s)) + h_2(s) x(s)], \tag{3.8}$$

for $1 \leq n \leq m$. From (3.8) and using the facts that $x(n) \leq y(n)$ for $1 \leq n \leq m$, we have

$$\begin{aligned} \Delta y(n) &= h_1(n) g(x(n)) + h_2(n) x(n) \\ &\leq h_1(n) g(y(n)) + h_2(n) y(n), \end{aligned}$$

i.e.

$$y(n+1) - [1 + h_2(n)] y(n) \leq h_1(n) g(y(n)), \tag{3.9}$$

for $1 \leq n \leq m$. Multiplying both sides of (3.9) by $\prod_{\sigma=1}^{n} [1 + h_2(\sigma)]^{-1}$, we obtain

$$\begin{aligned} & y(n+1) \prod_{\sigma=1}^{n} [1 + h_2(\sigma)]^{-1} - y(n) \prod_{\sigma=1}^{n-1} [1 + h_2(\sigma)]^{-1} \\ \leq \; & h_1(n) g(y(n)) \prod_{\sigma=1}^{n} [1 + h_2(\sigma)]^{-1}. \end{aligned} \tag{3.10}$$

By taking $n = s$ in (3.10) and summing up both sides of (3.10) from $s = 1$ to $n-1$, and using the fact that $y(1) = a(m)$, it follows that

$$\begin{aligned} & y(n) \prod_{\sigma=1}^{n-1} [1 + h_2(\sigma)]^{-1} - a(m) \\ \leq \; & \sum_{s=1}^{n-1} h_1(s) g(y(s)) \left(\prod_{\sigma=1}^{s} [1 + h_2(t)]^{-1} \right). \end{aligned} \tag{3.11}$$

From (3.11) we observe that

$$\begin{aligned} y(n) \leq \; & a(m) \prod_{\sigma=1}^{m-1} [1 + h_2(\sigma)] + \left(\prod_{\sigma=1}^{m-1} [1 + h_2(\sigma)] \right) \\ & \times \sum_{s=1}^{n-1} h_1(s) \left(\prod_{\sigma=1}^{s} [1 + h_2(\sigma)]^{-1} \right) g(y(s)), \end{aligned} \tag{3.12}$$

for $1 \le n \le m$. Now, an application of Lemma 1 yields

$$y(n) \le G^{-1}\left[G\left(a(m)\prod_{\sigma=1}^{m-1}[1+h_2(\sigma)]\right) + \left(\prod_{\sigma=1}^{m-1}[1+h_2(\sigma)]\right)\sum_{s=1}^{n-1}h_1(s)\left(\prod_{\sigma=1}^{s}[1+h_2(\sigma)]^{-1}\right)\right], \tag{3.13}$$

for $1 \le n \le m \le \alpha_1$. Using (3.13) in $x(n) \le y(n)$ we get

$$x(n) \le G^{-1}\left[G\left(a(m)\prod_{\sigma=1}^{m-1}[1+h_2(\sigma)]\right) + \left(\prod_{\sigma=1}^{m-1}[1+h_2(\sigma)]\right)\sum_{s=1}^{n-1}h_1(s)\left(\prod_{\sigma=1}^{s}[1+h_2(\sigma)]^{-1}\right)\right], \tag{3.14}$$

for $1 \le n \le m \le \alpha_1$. Since $m \in N$ was arbitrarily chosen, we deduce from (3.14) that

$$x(n) \le G^{-1}\left[G\left(a(n)\prod_{\sigma=1}^{n-1}[1+h_2(\sigma)]\right) + \left(\prod_{\sigma=1}^{n-1}[1+h_2(\sigma)]\right)\sum_{s=1}^{n-1}h_1(s)\left(\prod_{\sigma=1}^{s}[1+h_2(t)]^{-1}\right)\right], \tag{3.15}$$

for $1 \le n \le \alpha_1$. Now, by using (3.15) in $\frac{|u(n)|}{n} \le x(h)$, we get the desired bound in (2.3). This completes the proof of Theorem 1. ∎

The proof of Theorem 2 can be completed by following the same arguments as in the proof of Theorem 1 given above with suitable modification. Here we omit the details.

Remark 2. *From the proof of Theorem 1, we see that (3.3) and (3.5) yield* $|\Delta u(n)| \le x(n)$*. Hence by using the bound on* $x(n)$ *obtained in (3.15) in* $|\Delta u(n)| \le x(n)$*, we also get the bound on* $\Delta u(n)$ *for* $1 \le n \le \alpha_1$*.*

4. Further Extensions

In this section we extend the above method to obtain bounds on the solutions of the following third order difference equation

$$\Delta^3 u(n) + f\left(n, u(n), \Delta u(n), \Delta^2 u(n)\right) = 0, \tag{4.1}$$

with the given initial conditions

$$u(1) = d_0, \quad \Delta u(1) = d_1, \quad \Delta^2 u(1) = d_2, \tag{4.2}$$

where d_0, d_1, d_2 are constants and $f : N \times \mathbb{R}^3 - \mathbb{R}$ is a continuous function.

In the following theorem we establish the bound on the solution of (4.1) – (4.2).

Theorem 3. *Suppose that there exist functions* $h_i : N - \mathbb{R}_+$ $(i = 1, 2, 3)$ *such that*

$$|f(n, u, v, w)| \le h_1(n)(|u| + |w|) + h_2(n)\, g\left(\frac{|v|}{n}\right) + h_3(n), \tag{4.3}$$

where g is as defined in Theorem 1. Let G and G^{-1} be as defined in Theorem 1. Then for every solution $u(n)$ of (4.1) – (4.2) we have

$$|u(n)| \le |d_0| + \sum_{t=1}^{n-1} tH(t), \tag{4.4}$$

for $1 \le n \le \beta$, where

$$\begin{aligned} H(n) = G^{-1}\Bigg[G\left((|d_0| + b(n)) \prod_{\sigma=1}^{n-1} [1+\sigma+h_1(\sigma)]\right) \\ + \left(\prod_{\sigma=1}^{n-1} [1+\sigma+h_1(\sigma)]\right) \sum_{s=1}^{n-1} h_2(s) \left(\prod_{\sigma=1}^{s} [1+\sigma+h_1(\sigma)]^{-1}\right)\Bigg], \end{aligned} \tag{4.5}$$

in which $b(n)$ is defined by

$$b(n) = 1 + |d_1| + |d_2| + \sum_{s=1}^{n-1} h_3(s), \tag{4.6}$$

and

$$\begin{aligned} \beta &= \sup\Bigg\{n \in \mathbb{N} : G\left((|d_0| + b(n)) \prod_{\sigma=1}^{n-1} [1+\sigma+h_1(\sigma)]\right) \\ &\quad + \left(\prod_{\sigma=1}^{n-1} [1+\sigma+h_1(\sigma)]\right) \sum_{s=1}^{n-1} h_2(s) \\ &\quad \times \left(\prod_{\sigma=1}^{s} [1+\sigma+h_1(\sigma)]^{-1}\right) \in Dom\left(G^{-1}\right)\Bigg\}. \end{aligned}$$

Proof. Let $u(n)$ be a solution of (4.1) – (4.2). Setting $z(n) = \Delta u(n)$ and consequently $u(n) = d_0 + \sum_{t=1} z(t)$ in (4.1) – (4.2) reduces to

$$\Delta^2 z(n) + f\left(n, d_0 + \sum_{t=1}^{n-1} z(t), z(n), \Delta z(n)\right) = 0, \tag{4.7}$$

$$z(1) = d_1, \quad \Delta z(1) = d_2. \tag{4.8}$$

From (4.7) – (4.8) it is easy to observe that

$$\begin{aligned} \Delta z(n) &= d_2 - \sum_{s=1}^{n-1} f\left(s, d_0 + \sum_{t=1}^{s-1} z(t), z(s), \Delta z(s)\right), \\ z(1) &= d_1. \end{aligned} \tag{4.9}$$

The problem (4.9) is equivalent to

$$z(n) = d_1 + (n-1) d_2 - \sum_{s=1}^{n-1} (n-s) f\left(s, d_0 + \sum_{t=1}^{s-1} z(t), z(s), \Delta z(s)\right), \tag{4.10}$$

for $n \in N$. Using (4.3) in (4.9) and (4.10) we have

$$|\Delta z(n)| \leq |d_2| + \sum_{s=1}^{n-1}\left[h_1(s)\left(|d_0| + \sum_{t=1}^{s-1}|z(t)| + |\Delta z(s)|\right) + h_2(s)\, g\left(\frac{|z(s)|}{s}\right) + h_3(s)\right], \tag{4.11}$$

and

$$\frac{|z(n)|}{n} \leq |d_1| + |d_2| + \sum_{s=1}^{n-1}\left[h_1(s)\left(|d_0| + \sum_{t=1}^{s-1}|z(t)| + |\Delta z(s)|\right) + h_2(s)\, g\left(\frac{|z(s)|}{s}\right) + h_3(s)\right]. \tag{4.12}$$

Define a function $x(n)$ by

$$x(n) = |d_1| + |d_2| + \sum_{s=1}^{n-1}\left[h_1(s)\left(|d_0| + \sum_{t=1}^{s-1}|z(t)| + |\Delta z(s)|\right) + h_2(s)\, g\left(\frac{|z(s)|}{s}\right) + h_3(s)\right]. \tag{4.13}$$

Now, by using the facts that $|\Delta z(n)| \leq x(n)$ and $\frac{|z(n)|}{n} \leq x(n)$ in (4.13) we observe that

$$x(n) \leq b(n) + \sum_{s=1}^{n-1}\left[h_1(s)\left(|d_0| + \sum_{t=1}^{s-1} t x(t) + x(s)\right) + h_2(s)\, g(x(s))\right], \tag{4.14}$$

where $b(n)$ is defined by (4.6).

For a fixed element $m \in N$, from (4.14) we observe that

$$x(n) \leq b(m) + \sum_{s=1}^{n-1}\left[h_1(s)\left(|d_0| + \sum_{t=1}^{s-1} t x(t) + x(s)\right) + h_2(s)\, g(x(s))\right], \tag{4.15}$$

for $1 \leq n \leq m$. Define a function $y(n)$ by

$$y(n) = b(m) + \sum_{s=1}^{n-1}\left[h_1(s)\left(|d_0| + \sum_{t=1}^{s-1} t x(t) + x(s)\right) + h_2(s)\, g(x(s))\right], \tag{4.16}$$

for $1 \leq n \leq m$. from (4.16) and using the facts that $x(n) \leq y(n)$ for $1 \leq n \leq m$, we have

$$\begin{aligned} y(n) &= h_1(n)\left(|d_0| + \sum_{t=1}^{n-1} t x(t) + x(n)\right) + h_2(n)\, g(x(n)) \\ &\leq h_1(n)\left(|d_0| + \sum_{t=1}^{n-1} t y(t) + y(n)\right) + h_2(n)\, g(y(n)), \end{aligned} \tag{4.17}$$

for $1 \leq n \leq m$. Define a function $r(n)$ by

$$r(n) = |d_0| + \sum_{t=1}^{n-1} t y(t) + y(n), \tag{4.18}$$

for $1 \leq n \leq m$. From (4.18) and using the facts that

$$\Delta y(n) \leq h_1(n) r(n) + h_2(n) g(y(n))$$

from (4.17) and $y(n) \leq r(n)$ from (4.18) for $1 \leq n \leq m$ we have

$$\Delta r(n) = ny(n) + \Delta y(n) \leq [n + h_1(n)] r(n) + h_2(n) g(r(n)),$$

i.e.,

$$r(n+1) - [1 + n + h_1(n)] r(n) \leq h_2(n) g(r(n)), \tag{4.19}$$

for $1 \leq n \leq m$. Now by following similar arguments as in the proof of Theorem 1 below the inequality (3.9) and in view of the facts that $y(n) \leq r(n)$ and $x(n) \leq y(n)$, we get

$$x(n) \leq H(n), \tag{4.20}$$

for $1 \leq n \leq \beta$, where $H(n)$ is defined by (4.5) and β is as given in the theorem. Now, by using (4.20) in $\frac{|z(n)|}{n} \leq H(n)$, we have

$$|z(n)| \leq nH(n),$$

i.e.

$$|\Delta u(n)| \leq nH(n), \tag{4.21}$$

for $1 \leq n \leq \beta$. The inequality (4.21) implies that

$$|u(n+1)| - |u(n)| \leq |\Delta u(n)| \leq nH(n), \tag{4.22}$$

for $1 \leq n \leq \beta$. The inequality (4.22) implies the estimate

$$|u(n)| \leq |d_0| + \sum_{t=1}^{n-1} tH(t),$$

for $1 \leq n \leq \beta$. The proof is thus complete. ∎

Remark 3. *We note that if we replace the condition (4.3) in Theorem 3 by*

$$|f(n, u, v, w)| \leq h_1(n) \left(|u| + \frac{|v|}{n} \right) + h_2(n) g(|w|) + h_3(n),$$

then by following exactly the same argument as in the proof of Theorem 3, we can obtain the bound on the solution of (4.1) – (4.2).

5. Some Generalisations

In this section we indicate that the method employed in earlier sections can be used to obtain bounds on the solutions of the more general sum-difference equation of the form

$$\Delta^2 u(n) + f\left(n, u(n), \Delta u(n), \sum_{s=1}^{n-1} k(n, s, u(s), \Delta u(s))\right) = 0, \tag{5.1}$$

with the given initial conditions

$$u(1) = c_1, \quad \Delta u(1) = c_2, \tag{5.2}$$

where c_1, c_2 are constants and $k : N^2 \times \mathbb{R}^2 \to \mathbb{R}$, $f : N \times \mathbb{R}^3 \to \mathbb{R}$ are continuous.

The following theorem gives the bound on the solution of (5.1) – (5.2) under some suitable conditions on the functions f and k involved in (5.1).

Theorem 4. *Suppose that there exists functions h, $h_i : N \to \mathbb{R}_+$, $(i = 1, 2, 3, 4)$ such that*

$$
\begin{aligned}
&(5.3) \qquad |f(n, u, v, w)| \leq h_1(n)\, g\left(\frac{|u|}{n}\right) + h_2(n)\,(|v| + |w|) + h(n), \\
&(5.4) \qquad |k(n, s, u, v)| \leq h_3(s)\, g\left(\frac{|u|}{s}\right) + h_4(s)\,|v|,
\end{aligned}
$$

where g is as defined in Theorem 1. Let G and G^{-1} be as defined in Theorem 1. Then for every solution $u(n)$ of (5.1) – (5.2) we have

$$
\begin{aligned}
(5.5) \qquad |u(n)| \leq\ & nG^{-1}\Bigg[G\left(c(n)\prod_{t=1}^{n-1}[1 + h_2(t) + h_4(t)]\right) \\
& + \left(\prod_{t=1}^{n-1}[1 + h_2(t) + h_4(t)]\right)\sum_{s=1}^{n-1}[h_1(s) + h_3(s)] \\
& \times \left(\prod_{t=1}^{s}[1 + h_2(t) + h_4(t)]^{-1}\right)\Bigg],
\end{aligned}
$$

for $1 \leq n \leq \gamma$, where

$$
c(n) = 1 + |c_1| + |c_2| + \sum_{s=1}^{n-1} h(s),
$$

and

$$
\begin{aligned}
\gamma = \sup\Bigg\{ & n \in \mathbb{N} : G\left(c(n)\prod_{t=1}^{n-1}[1 + h_2(t) + h_4(t)]\right) \\
& + \left(\prod_{t=1}^{n-1}[1 + h_2(t) + h_4(t)]\right)\sum_{s=1}^{n-1}[h_1(s) + h_3(s)] \\
& \times \left(\prod_{t=1}^{s}[1 + h_2(t) + h_4(t)]^{-1}\right) \in Dom\left(G^{-1}\right)\Bigg\}.
\end{aligned}
$$

The proof of this theorem can be completed by following the proofs of Theorems 1 and 3 with suitable modifications. Here we omit the details.

Remark 4. *We note that if we replace the conditions (5.3) and (5.4) by*

$$
\begin{aligned}
|f(n, u, v, w)| &\leq h_1(n)\left(\frac{|u|}{n} + |w|\right) + h_2(n)\, g(|v|) + h(n), \\
|k(n, s, u, v)| &\leq h_3(s)\, g\left(\frac{|u|}{s}\right) + h_4(s)\, g(v),
\end{aligned}
$$

then the bound obtained in (5.5) reduces to

$$|u(n)| \leq nG^{-1}\left[G\left(c(n)\prod_{t=1}^{n-1}[1+h_1(t)+h_3(t)]\right) + \left(\prod_{t=1}^{n-1}[1+h_1(t)+h_3(t)]\right)\sum_{s=1}^{n-1}[h_2(s)+h_4(s)] \times \left(\prod_{t=1}^{s}[1+h_1(t)+h_3(t)]^{-1}\right)\right], \tag{5.6}$$

for $1 \leq n \leq \gamma_1$*, where* γ_1 *is defined so that the existence of the right part of (5.6) should be assured.*

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