SOME BOUNDS IN TERMS OF $\Delta-$SEMINORMS FOR OSTROWSKI-GRÜSS TYPE INEQUALITIES

P. CERONE AND S.S. DRAGOMIR

Abstract. In this paper we point out some bounds for the remainder of a generalised Ostrowski type formula by the use of $\Delta-$seminorms.

1. Introduction

As in [1], let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two sequences of harmonic polynomials, that is, polynomials satisfying

$$P_n'(t) = P_{n-1}(t), \quad P_0(t) = 1, \quad t \in \mathbb{R}, \tag{1.1}$$

$$Q_n'(t) = Q_{n-1}(t), \quad Q_0(t) = 1, \quad t \in \mathbb{R}. \tag{1.2}$$

In [1], the authors proved the following result.

Lemma 1. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two harmonic polynomials. Set*

$$S_n(t,x) := \begin{cases} P_n(t), & t \in [a,x] \\ Q_n(t), & t \in (x,b] \end{cases}, \quad (t,x) \in [a,b]^2. \tag{1.3}$$

Then we have the equality

$$\begin{aligned} &\int_a^b f(t)\,dt \\ &= \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ &\quad - P_k(a) f^{(k-1)}(a) \Big] + (-1)^n \int_a^b S_n(t,x) f^{(n)}(t)\,dt, \end{aligned} \tag{1.4}$$

provided that $f : [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$.

Using the following "pre-Grüss" inequality

$$|T(f,g)| \le \frac{1}{2}\sqrt{T(f,f)}\,(\Gamma - \gamma), \tag{1.5}$$

where

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x) g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

is the Chebychev functional and f, g are such that the previous integrals exist and $\gamma \le g(x) \le \Gamma$ for a.e. $x \in [a,b]$, the authors of [1] proved basically the

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following inequality for estimating the integral $\int_a^b f(t)\,dt$ in terms of the harmonic polynomials $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$.

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is integrable and $\gamma_n \leq f^{(n)} \leq \Gamma_n$ for all $t\in[a,b]$. Put*

$$U_n(x) := \frac{1}{b-a}\left[Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)\right]. \tag{1.6}$$

Then for all $x\in[a,b]$, we have the inequality

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b) + (P_k(x)-Q_k(x))\,f^{(k-1)}(x)\right.\\ &\left. -P_k(a)\,f^{(k-1)}(a)\Big] - (-1)^n U_n(x)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]\right|\\ \leq\ & \frac{1}{2}K(x)(\Gamma_n-\gamma_n)(b-a), \end{aligned} \tag{1.7}$$

where

$$K(x) := \left\{\frac{1}{b-a}\int_a^x P_n^2(t)\,dt + \int_x^b Q_n^2(t)\,dt - [U_n(x)]^2\right\}^{\frac{1}{2}}. \tag{1.8}$$

A number of particular cases that were obtained by an appropriate choice of harmonic polynomials have also been presented in [1].

In the recent paper [2], Dragomir proved the following refinement of (1.7).

Theorem 2. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$ $(n\geq 1)$. If we denote*

$$\left[f^{(n-1)};a,b\right] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b) + (P_k(x)-Q_k(x))\,f^{(k-1)}(x)\right.\\ &-P_k(a)\,f^{(k-1)}(a)\Big] - (-1)^n\left[Q_{n+1}(b) - Q_{n+1}(x)\right.\\ &\left.+P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\Big|\\ \leq\ & K(x)(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}}\\ &\left(\leq \frac{1}{2}K(x)(b-a)(\Gamma_n-\gamma_n)\quad \text{if}\ \ f^{(n)}\in L_\infty(a,b)\right), \end{aligned} \tag{1.9}$$

for all $x\in[a,b]$ and $K(x)$ as is given in (1.8).

2. Some Preliminary Results Involving Lebesgue Norms on $[a,b]^2$

For $f\in L_p[a,b]$ $(p\in[1,\infty))$, we can define the functional (see also [3])

$$\|f\|_p^{\Delta} := \left(\int_a^b\int_a^b |f(t)-f(s)|^p\,dtds\right)^{\frac{1}{p}} \tag{2.1}$$

and for $f\in L_{\infty}[a,b]$, we can define

$$\left\Vert f\right\Vert _{\infty}^{\Delta}:=ess\sup_{(t,s)\in[a,b]^{2}}\left\vert f(t)-f(s)\right\vert . \tag{2.2}$$

If we consider $f_{\Delta}:[a,b]^{2}\rightarrow\mathbb{R}$,

$$f_{\Delta}(t,s)=f(t)-f(s), \tag{2.3}$$

then, obviously

$$\left\Vert f\right\Vert _{p}^{\Delta}=\left\Vert f_{\Delta}\right\Vert _{p},\ \ p\in[1,\infty], \tag{2.4}$$

where $\left\Vert \cdot\right\Vert _{p}$ are the usual Lebesque p-norms on $[a,b]^{2}$.

Using the properties of the Lebesque $p-$norms, we may deduce the following semi-norm properties for $\left\Vert \cdot\right\Vert _{p}^{\Delta}$:

- (i) $\left\Vert f\right\Vert _{p}^{\Delta}\geq0$ for $f\in L_{p}[a,b]$ and $\left\Vert f\right\Vert _{p}^{\Delta}=0$ implies that $f=c$ (c is a constant) a.e. in $[a,b]$;
- (ii) $\left\Vert f+g\right\Vert _{p}^{\Delta}\leq\left\Vert f\right\Vert _{p}^{\Delta}+\left\Vert g\right\Vert _{p}^{\Delta}$ if $f,g\in L_{p}[a,b]$;
- (iii) $\left\Vert \alpha f\right\Vert _{p}^{\Delta}=\left\vert \alpha\right\vert \left\Vert f\right\Vert _{p}^{\Delta}$.

We note that if $p=2$, then,

$$\begin{aligned}\left\Vert f\right\Vert _{2}^{\Delta} & =\left(\int_{a}^{b}\int_{a}^{b}\left(f(t)-f(s)\right)^{2}dtds\right)^{\frac{1}{2}}\\ & =\sqrt{2}\left[(b-a)\left\Vert f\right\Vert _{2}^{2}-\left(\int_{a}^{b}f(t)\,dt\right)^{2}\right]^{\frac{1}{2}}.\end{aligned}$$

If $f:[a,b]\rightarrow\mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\left\Vert f\right\Vert _{p}^{\Delta}$ in terms of $\left\Vert f'\right\Vert _{p}$.

Theorem 3. *Assume that $f:[a,b]\rightarrow\mathbb{R}$ is absolutely continuous on $[a,b]$.*

(i) If $p\in[1,\infty)$, then we have the inequality

$$\left\Vert f\right\Vert _{p}^{\Delta}\leq\begin{cases}\dfrac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\left\Vert f'\right\Vert _{\infty} & if\quad f'\in L_{\infty}[a,b];\\ \dfrac{\left(2\beta^{2}\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\left\Vert f'\right\Vert _{\alpha} & if\quad f'\in L_{\alpha}[a,b],\\ & \qquad\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ (b-a)^{\frac{2}{p}}\left\Vert f'\right\Vert _{1} & if\quad f'\in L_{1}[a,b],\end{cases} \tag{2.5}$$

(ii) If $p=\infty$, then we have the inequality

$$\left\Vert f\right\Vert _{\infty}^{\Delta}\leq\begin{cases}(b-a)\left\Vert f'\right\Vert _{\infty} & if\quad f'\in L_{\infty}[a,b];\\ (b-a)^{\frac{1}{\beta}}\left\Vert f'\right\Vert _{\alpha} & if\quad f'\in L_{\alpha}[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \left\Vert f'\right\Vert _{1}.\end{cases} \tag{2.6}$$

Proof. As $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then $f(t)-f(s)=\int_s^t f'(u)\,du$ for all $t,s\in[a,b]$, and then

$$\begin{aligned}(2.7)\quad & |f(t)-f(s)| \\ &= \left|\int_s^t f'(u)\,du\right| \leq \begin{cases} |t-s|\,\|f'\|_\infty & \text{if } \ f'\in L_\infty[a,b]; \\ |t-s|^{\frac{1}{\beta}}\,\|f'\|_\alpha & \text{if } \ f'\in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1 & \text{if } \ f'\in L_1[a,b]\end{cases}\end{aligned}$$

and so for $p\in[1,\infty)$, we may write

$$|f(t)-f(s)|^p \leq \begin{cases} |t-s|^p\,\|f'\|_\infty^p & \text{if } \ f'\in L_\infty[a,b]; \\ |t-s|^{\frac{p}{\beta}}\,\|f'\|_\alpha^p & \text{if } \ f'\in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1^p & \text{if } \ f'\in L_1[a,b],\end{cases}$$

and then from (2.3), (2.4)

$$\|f\|_p^\Delta \leq \begin{cases} \|f'\|_\infty\left(\int_a^b\int_a^b|t-s|^p\,dtds\right)^{\frac{1}{p}} & \text{if } \ f'\in L_\infty[a,b]; \\ \|f'\|_\alpha\left(\int_a^b\int_a^b|t-s|^{\frac{p}{\beta}}\,dtds\right)^{\frac{1}{p}} & \text{if } \ f'\in L_\alpha[a,b], \\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1\left(\int_a^b\int_a^b dtds\right)^{\frac{1}{p}} & \text{if } \ f'\in L_1[a,b]. \end{cases} \tag{2.8}$$

Further, since

$$\begin{aligned}\left(\int_a^b\int_a^b|t-s|^p\,dtds\right)^{\frac{1}{p}} &= \left[\int_a^b\left(\int_a^t(t-s)^p\,ds+\int_t^b(s-t)^p\,ds\right)dt\right]^{\frac{1}{p}} \\ &= \left(\int_a^b\left[\frac{(t-a)^{p+1}+(b-t)^{p+1}}{p+1}\right]dt\right)^{\frac{1}{p}} \\ &= \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}},\end{aligned}$$

giving

$$\left(\int_a^b\int_a^b|t-s|^{\frac{p}{\beta}}\,dtds\right)^{\frac{1}{p}} = \frac{\left(2\beta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}},$$

and

$$\left(\int_a^b\int_a^b dtds\right)^{\frac{1}{p}} = (b-a)^{\frac{2}{p}},$$

we obtain, from (2.8), the stated result (2.5).

Using (2.7) we have (for $p=\infty$) that

$$\|f\|_{\infty}^{\Delta} \leq \begin{cases} \|f'\|_{\infty} \, ess \sup\limits_{(t,s)\in[a,b]^2} |t-s| \\ \\ \|f'\|_{\alpha} \, ess \sup\limits_{(t,s)\in[a,b]} |t-s|^{\frac{1}{\beta}} \\ \\ \|f'\|_{1} \end{cases} = \begin{cases} (b-a)\|f'\|_{\infty} \\ \\ (b-a)^{\frac{1}{\beta}} \|f'\|_{\alpha} \\ \\ \|f'\|_{1} \end{cases}$$

and the inequality (2.6) is also proved. ∎

3. Some Bounds in Terms of $\Delta-$Seminorms

We start with the following result which obtains bounds for the left hand side of (1.9) (or equivalently (1.7)) in terms of the $\Delta-$seminorms of the previous section.

Theorem 4. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two harmonic polynomials. Set $S_n(\cdot,\cdot)$ as in Lemma 1 and assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned} &\Bigg|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b) f^{(k-1)}(b) + (P_k(x)-Q_k(x)) f^{(k-1)}(x) \\ &-P_k(a) f^{(k-1)}(a)\Big] - (-1)^n\left[Q_{n+1}(b) - Q_{n+1}(x)\right. \\ &\left.+P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\Bigg| \\ \leq\ & \frac{1}{2(b-a)} \times \begin{cases} \|S_n(\cdot,x)\|_1^{\Delta} \left\|f^{(n)}\right\|_{\infty}^{\Delta} & \text{if } \ f^{(n)}\in L_\infty[a,b]; \\ \\ \|S_n(\cdot,x)\|_q^{\Delta} \left\|f^{(n)}\right\|_{p}^{\Delta} & \text{if } \ \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ & \quad f^{(n)}\in L_p[a,b]; \\ \|S_n(\cdot,x)\|_{\infty}^{\Delta} \left\|f^{(n)}\right\|_{1}^{\Delta} \end{cases} \end{aligned} \tag{3.1}$$

and the $\Delta-$seminorms $\|\cdot\|_p^{\Delta}$ $(p\in[1,\infty])$ are defined as in Section 2.

Proof. Recall Korkine's identity

$$T(h,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (h(t)-h(s))(g(t)-g(s))\,dtds, \tag{3.2}$$

where $T(\cdot,\cdot)$ is the Chebychev functional. That is, we recall that

$$T(h,g) = \frac{1}{b-a}\int_a^b h(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b h(x)\,dx \cdot \int_a^b g(x)\,dx,$$

provided that all the involved integrals exist.

Using (3.2) and the identity (1.4), gives (see also [1]):

$$\begin{aligned} &\int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ &-P_k(a) f^{(k-1)}(a)\Big] - (-1)^n \left[Q_{n+1}(b) - Q_{n+1}(x)\right. \\ &\left.+P_{n+1}(x) - P_{n+1}(a)\right] \left[f^{(n-1)}; a, b\right] \\ = &\frac{1}{2(b-a)^2} \int_a^b \int_a^b (S_n(t,x) - S_n(s,x)) (f^{(n)}(t) - f^{(n)}(s)) dtds. \end{aligned} \tag{3.3}$$

Using Hölder's integral inequality for double integrals, we may write

$$\begin{aligned} B &:= \left| \int_a^b \int_a^b (S_n(t,x) - S_n(s,x)) (f^{(n)}(t) - f^{(n)}(s)) dtds \right| \\ &\leq \left(\int_a^b \int_a^b |S_n(t,x) - S_n(s,x)|^q \, dtds \right)^{\frac{1}{q}} \left(\int_a^b \int_a^b \left| f^{(n)}(t) - f^{(n)}(s) \right|^p dtds \right)^{\frac{1}{p}} \\ &= \|S_n(\cdot,x)\|_q^{\Delta} \left\| f^{(n)} \right\|_p^{\Delta}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

If $q = 1$, we have

$$B \leq \|S_n(\cdot,x)\|_1^{\Delta} \left\| f^{(n)} \right\|_{\infty}^{\Delta}$$

and if $p = 1$, then

$$B \leq \|S_n(\cdot,x)\|_{\infty}^{\Delta} \left\| f^{(n)} \right\|_1^{\Delta}.$$

Further, using the identity (3.3) and the properties of modulus, we obtain (3.1). ∎

Remark 1. *For $p = q = 2$, we recapture Theorem 2 and so Theorem 4 represents an Ostrowski-Grüss result whose bounds are given in terms of the $\Delta-$seminorms whose properties are given in Section 2.*

The following corollary holds.

Corollary 1. *Let $\{P_n\}_{n\in\mathbb{N}}$, $\{Q_n\}_{n\in\mathbb{N}}$ and $\{S_n\}_{n\in\mathbb{N}}$ be as in Theorem 4. If $f : [a,b] \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous, then we have the inequality:*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right. \\ &-P_k(a) f^{(k-1)}(a)\Big] - (-1)^n \left[Q_{n+1}(b) - Q_{n+1}(x)\right. \\ &\left.\left.+P_{n+1}(x) - P_{n+1}(a)\right] \left[f^{(n-1)}; a, b\right] \right| \end{aligned} \tag{3.4}$$

$$\leq \begin{cases} \frac{1}{2}\|S_n(\cdot,x)\|_1^{\Delta}\left\|f^{(n+1)}\right\|_{\infty} & \text{if} \quad f^{(n+1)} \in L_{\infty}[a,b]; \\ \frac{1}{2}(b-a)^{\frac{1}{\beta}-1}\|S_n(\cdot,x)\|_1^{\Delta}\left\|f^{(n+1)}\right\|_{\alpha} & \text{if} \quad f^{(n+1)} \in L_{\alpha}[a,b], \\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{1}{2(b-a)}\|S_n(\cdot,x)\|_1^{\Delta}\left\|f^{(n+1)}\right\|_1 & \\ \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\|S_n(\cdot,x)\|_q^{\Delta}\left\|f^{(n+1)}\right\|_{\infty} & \text{if} \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ & \quad f^{(n+1)} \in L_{\alpha}[a,b]; \\ \frac{2^{\frac{1}{p}-1}\left(\beta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}-1}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\|S_n(\cdot,x)\|_q^{\Delta}\left\|f^{(n+1)}\right\|_{\alpha} & \text{if} \quad f^{(n+1)} \in L_{\alpha}[a,b], \\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}(b-a)^{\frac{2}{p}-1}\|S_n(\cdot,x)\|_q^{\Delta}\left\|f^{(n+1)}\right\|_1 & \text{if} \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{(b-a)^2}{6}\|S_n(\cdot,x)\|_{\infty}^{\Delta}\left\|f^{(n+1)}\right\|_{\infty} & \text{if} \quad f^{(n+1)} \in L_{\infty}[a,b]; \\ \frac{\beta^2(b-a)^{\frac{1}{\beta}+1}}{2(\beta+1)(\beta+2)}\|S_n(\cdot,x)\|_{\infty}^{\Delta}\left\|f^{(n+1)}\right\|_{\alpha} & \text{if} \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & \quad f^{(n+1)} \in L_{\alpha}[a,b]; \\ \frac{1}{2}(b-a)\|S_n(\cdot,x)\|_{\infty}^{\Delta}\left\|f^{(n+1)}\right\|_1. & \end{cases}$$

The proof follows by Theorem 4 and by Theorem 3 (Section 2) applied for the $\Delta-$seminorm of the mapping $f^{(n)}$. We omit the details.

Remark 2. *If we choose*

$$S_n(t,x) = \begin{cases} P_n(t), & t \in [a,x], \\ kH_n(t-x)+P_n(t), & t \in (x,b] \end{cases} \tag{3.5}$$

with $P_n(\cdot)$ and $H_n(\cdot - x)$ harmonic polynomials satifying (1.1) and $H_n(0) = 0$ for all $n \in \mathbb{N}$ then, $S_n(\cdot,x)$ is absolutely continuous on $[a,b]$. Hence, the bounds obtained in Theorem 3 will hold for $\|S_n(\cdot,x)\|_q$, $q \geq 1$ in terms of $\|S_n'(\cdot,x)\|_{\gamma}$, $\gamma \geq 1$. Here,

$$S_n'(t,x) = \begin{cases} P_{n-1}(t), & t \in [a,x], \\ kH_{n-1}(t-x)+P_{n-1}(t), & t \in (x,b], \end{cases} \tag{3.6}$$

where the differentiation is with respect to t within each of the two subintervals. With the above development, we have from Theorem 3:

(*i*) *For $q \in [1,\infty)$,*

$$\|S_n(\cdot,x)\|_q^{\Delta} \leq \begin{cases} 2^{\frac{1}{q}}(b-a)^{1+\frac{2}{q}}\|S_n'(\cdot,x)\|_{\infty}, & S_n' \in L_{\infty}[a,b]; \\ \frac{\left(2\delta^2\right)^{\frac{1}{q}}(b-a)^{\frac{1}{\delta}+\frac{2}{q}}}{(q+\delta)(q+2\delta)^{\frac{1}{q}}}\|S_n'(\cdot,x)\|_{\gamma}, & S_n' \in L_{\gamma}[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ (b-a)^{\frac{2}{q}}\|S_n'(\cdot,x)\|_1, & S_n' \in L_1[a,b], \end{cases} \tag{3.7}$$

(ii)

$$\|S_n(\cdot,x)\|_\infty^\Delta \le \begin{cases} (b-a)\|S_n'(\cdot,x)\|_\infty, & S_n' \in L_\infty[a,b]; \\ (b-a)^{\frac{1}{\delta}}\|S_n'(\cdot,x)\|_\gamma, & S_n' \in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|S_n'(\cdot,x)\|_1, & S_n' \in L_1[a,b], \end{cases} \tag{3.8}$$

and so substitution into the right hand side of Corollary 1 would give bounds involving 27 branches.

Corollary 2. *Let* $\{P_n\}_{n\in\mathbb{N}}$ *be as in Theorem 4. Then for* $f:[a,b]\to\mathbb{R}$ *and* $f^{(n)}$ *absolutely continuous, we have*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)\right]\right. \\ &\left. -(-1)^n\left[P_{n+1}(b)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\right| \end{aligned} \tag{3.9}$$

$$\le \begin{cases} \frac{1}{2}\|P_n\|_1^\Delta\left\|f^{(n+1)}\right\|_\infty & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\ \frac{1}{2}(b-a)^{\frac{1}{\beta}-1}\|P_n\|_1^\Delta\left\|f^{(n+1)}\right\|_\alpha & \text{if } f^{(n+1)}\in L_\alpha[a,b], \\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{1}{2(b-a)}\|P_n\|_1^\Delta\left\|f^{(n+1)}\right\|_1 & \\ \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\|P_n\|_q^\Delta\left\|f^{(n+1)}\right\|_\infty & \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ & \quad f^{(n+1)}\in L_\alpha[a,b]; \\ \frac{2^{\frac{1}{p}-1}\left(\beta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}-1}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\|P_n\|_q^\Delta\left\|f^{(n+1)}\right\|_\alpha & \text{if } f^{(n+1)}\in L_\alpha[a,b], \\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}(b-a)^{\frac{2}{p}-1}\|P_n\|_q^\Delta\left\|f^{(n+1)}\right\|_1 & \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{(b-a)^2}{6}\|P_n\|_\infty^\Delta\left\|f^{(n+1)}\right\|_\infty & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\ \frac{\beta^2(b-a)^{\frac{1}{\beta}+1}}{2(\beta+1)(\beta+2)}\|P_n\|_\infty^\Delta\left\|f^{(n+1)}\right\|_\alpha & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & \quad f^{(n+1)}\in L_\alpha[a,b]; \\ \frac{1}{2}(b-a)\|P_n\|_\infty^\Delta\left\|f^{(n+1)}\right\|_1. & \end{cases}$$

where for $q\in[1,\infty)$

$$\|P_n\|_q^\Delta \le \begin{cases} 2^{\frac{1}{q}}(b-a)^{1+\frac{2}{q}}\|P_{n-1}\|_\infty, & P_n\in L_\infty[a,b]; \\ \left(2\delta^2\right)^{\frac{1}{q}}(b-a)^{\frac{1}{\delta}+\frac{2}{q}}\|P_{n-1}\|_\gamma, & P_n\in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ (b-a)^{\frac{2}{q}}\|P_{n-1}\|_1, & P_n\in L_1[a,b], \end{cases} \tag{3.10}$$

and

$$\|P_n\|_{\infty}^{\Delta} \leq \begin{cases} (b-a)\|P_{n-1}\|_{\infty}, & P_n \in L_{\infty}[a,b]; \\ (b-a)^{\frac{1}{\delta}}\|P_{n-1}\|_{\gamma}, & P_n \in L_{\gamma}[a,b], \\ & \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|P_{n-1}\|_{1}, & P_n \in L_1[a,b]. \end{cases} \tag{3.11}$$

Proof. Taking $k = 0$ in (3.5) or equivalently $Q_n(t) \equiv P_n(t)$ in (1.3) gives from Corollary 1 the stated results where we have used (3.6) – (3.8). ∎

Remark 3. *As an example, take*

$$\tilde{P}_n(t) = \frac{(t-\theta)^n}{n!} \tag{3.12}$$

with

$$\theta = (1-\lambda)a + \lambda b, \quad \lambda \in [0,1],$$

then,

$$\begin{aligned} \left\|\tilde{P}_n\right\|_{\infty} &= \sup_{t\in[a,b]}\left|\tilde{P}_n(t)\right| = \frac{(b-a)^n}{n!}\max\{\lambda^n,(1-\lambda)^n\} \\ &= \frac{(b-a)^n}{n!}\left[\frac{1}{2}|\left|\lambda-\frac{1}{2}\right|\right]^n. \end{aligned} \tag{3.13}$$

Further,

$$\begin{aligned} \left\|\tilde{P}_n\right\|_{\gamma} &= \left(\int_a^b \left|\tilde{P}_n(t)\right|^{\gamma} dt\right)^{\frac{1}{\gamma}} \qquad (\gamma \in [1,\infty)) \\ &= \frac{1}{n!}\left[\int_a^{\theta}(\theta-t)^{n\gamma}dt + \int_{\theta}^{b}(t-\theta)^{n\gamma}dt\right]^{\frac{1}{\gamma}} \\ &= \frac{1}{n!}\left[\frac{(\theta-a)^{n\gamma+1}+(b-\theta)^{n\gamma+1}}{n\gamma+1}\right]^{\frac{1}{\gamma}}, \end{aligned}$$

giving

$$\left\|\tilde{P}_n\right\|_{\gamma} = \frac{(b-a)^{n+\frac{1}{\gamma}}}{n!}\left[\frac{\lambda^{n\gamma+1}+(1-\lambda)^{n\gamma+1}}{n\gamma+1}\right]^{\frac{1}{\gamma}}. \tag{3.14}$$

Thus, from (3.9) with $P_n(t)$ as given by (3.12) gives

$$\begin{aligned} \tau_\lambda \quad : &= \Bigg|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\frac{(b-a)^k}{k!} \\ &\times\left[(1-\lambda)^k f^{(k-1)}(b) - \lambda^k f^{(k-1)}(a)\right] - (-1)^n\frac{(b-a)^{n+1}}{(n+1)!} \\ &\times\left[(1-\lambda)^{n+1}f^{(n)}(b) - \lambda^{n+1}f^{(n)}(a)\right]\left[f^{(n-1)};a,b\right]\Bigg| \\ \leq\ & B\left(\left\|\tilde{P}_n\right\|^{\Delta}\right), \end{aligned} \tag{3.15}$$

where $B\left(\|P_n\|^{\Delta}\right)$ *is the right hand side of (3.9) with* $\left\|\tilde{P}_n\right\|_q^{\Delta}$, $\left\|\tilde{P}_n\right\|_{\infty}^{\Delta}$ *given by (3.10), (3.11) on using (3.13) and (3.14).*

For $\lambda = \frac{1}{2}$ *the left hand side of (3.15) simplifies to*

$$\begin{aligned}\frac{1}{b-a}\tau_{\frac{1}{2}} &= \left|\frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}\left(\frac{b-a}{2}\right)^k\left[f^{(k-1)};a,b\right]\right.\\ &\quad \left. -\frac{(-1)^n}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+1}\left[f^{(n)};a,b\right]\times\left[f^{(n-1)};a,b\right]\right|,\end{aligned}$$

where $[g;a,b] = \frac{g(b)-g(a)}{b-a}$.

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: pc@matilda.vu.edu.au
URL: http://sci.vu.edu.au/~pc

E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html