ON CERTAIN PARTIAL INTEGRAL INEQUALITIES FOR NON-SELF-ADJOINT HYPERBOLIC PARTIAL DIFFERENTIAL AND INTEGRAL EQUATIONS

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Abstract. The aim of this paper is to establish some new partial integral inequalities in two independent variables which can be used in the study of qualitative behaviour of the solutions of various classes of nonlinear non-self-adjoint hyperbolic partial differential and integral equations as ready and powerful tools.

1. Introduction

The theory of partial differential and integral equations is in a process of continual development and it has become significant for its various applications. One of the most useful techniques used in the qualitative behaviour of the solutions of partial differential and integral equations consists in applying some kinds of partial differential and integral inequalities and variational principles involving functions and their derivatives. During the past few years many authors (see [1] - [14], [17] - [22]) have established several partial differential and integral inequalities in two or more independent variables which can be used in the analysis of various problems in the theory of hyperbolic partial differential and integral equations. However, the lack of suitable nonlinear partial integral inequalities prevents us from studying the qualitative behaviour of solutions of various classes of nonlinear non-self-adjoint hyperbolic partial differential and integral equations under less restrictive assumptions on the functions. Our objective here is to present a number of new partial integral inequalities in two independent variables which can be used as handy tools in the qualitative behaviour of the solutions of several classes of nonlinear non-self-adjoint hyperbolic partial differential and integral equations.

2. Partial Integral Inequalities

In this section we state and prove some new partial integral inequalities in two independent variables which can be used in the analysis of various problems in the theory of nonlinear non-self-adjoint hyperbolic partial differential and integral equations. We use the following assumptions in our subsequent discussion.

(H_1) $a(x, y)$ is a real-valued, positive, continuous function defined on a domain D and nondecreasing in both variables,

(H_2) $u(x, y)$, $b(x, y)$, $c(x, y)$ and $p(x, y)$ are real-valued non-negative continuous functions defined on D,

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(H$_3$) $u(x,y)$, $b(x,y)$, $c(x,y)$ and $q(x,y)$ are real-valued non-negative continuous functions defined on D,

(H$_4$) $G(u(x,y))$ is a real-valued, positive, continuous, monotonic, nondecreasing, subadditive and submultiplicative function for $u(x,y)\geq 0$, $(x,y)\in D$ and $G_y(u(x,y))=\frac{\partial}{\partial y}G(u(x,y))\geq 0$ for $u(x,y)\geq 0$, $(x,y)\in D$,

(H$_5$) $H(u(x,y))$ is a real-valued, positive, continuous, monotonic, nondecreasing function for $u(x,y)\geq 0$, $(x,y)\in D$,

(H$_6$) $K(x,y,s,t,u(x,y))$ and $W(x,y,u(x,y))$ are real-valued non-negative continuous functions defined on $D^2X\mathbb{R}^+$ and $DX\mathbb{R}^+$ respectively (where $\mathbb{R}^+$ is the set of non-negative real numbers) and nondecreasing in the last variables, and $K(x,y,s,t,u)$ is uniformly Lipschitz in the last variable,

(H$_7$) $u(x,y)$, $b(x,y)$, $c(x,y)$, $p(x,y)$, $r(x,y)$, $h(x,y)$ and $f(x,y)$ are real-valued non-negative continuous functions on a domain D,

(H$_8$) $u(x,y)$, $b(x,y)$, $c(x,y)$, $q(x,y)$, $r(x,y)$, $h(x,y)$ and $f(x,y)$ are real-valued non-negative continuous functions on a domain D,

(H$_9$) $P_0(x_0,y_0)$ and $P(x,y)$ are two points in D such that $(x-x_0)(y-y_0)\geq 0$ and R the rectangular region whose opposite corners are the points P_0 and P, (see Figure 3 in [11]),

(H$_{10}$) The functions $v(s,t;x,y)$ and $e(s,t;x,y)$ are the Riemann functions for the partial differential operators L and T respectively and satisfy all the properties of Riemann functions for operators with continuous coefficient.

A useful partial integral inequality is established in the following theorem.

Theorem 1. *Suppose (H$_1$), (H$_2$), (H$_4$) and (H$_5$) are true. If*

$$u(x,y) \leq a(x,y)+p(x,y)\left(\int_{x_0}^{x} b(s,y)\,u(s,y)\,ds\right) + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(u(s,t))\,dsdt\right), \tag{2.1}$$

for all $(x,y)\in D$, then for all $(x,y)\in D_1\subset D$,

$$\begin{aligned} &u(x,y) \\ &\leq F(x,y)\left[a(x,y)+H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,F(s,t))\,dsdt\right) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(F(s,t))\,dsdt\right]\right)\right], \end{aligned} \tag{2.2}$$

where

$$F(x,y)=1+p(x,y)\left(\int_{x_0}^{x} b(s,y)\exp\left(\int_{s}^{x} b(\xi,y)\,p(\xi,y)\,d\xi\right)ds\right), \tag{2.3}$$

$$\Omega(r)=\int_{r_0}^{r}\frac{ds}{G(H(s))},\quad r\geq r_0>0, \tag{2.4}$$

Ω^{-1} is the inverse function of Ω, and

$$\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,F(s,t))\,dsdt\right) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(F(s,t))\,dsdt \in Dom\left(\Omega^{-1}\right)$$

for all $(x,y) \in D_1$.

Proof. Define

$$m(x,y) = a(x,y) + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(u(s,t))\, dsdt\right), \tag{2.5}$$

then (2.1) can be restated as

$$u(x,y) \le m(x,y) + p(x,y)\left(\int_{x_0}^{x} b(s,y)\, u(s,y)\, ds\right). \tag{2.6}$$

Since $m(x,y)$ is positive, monotonic and nondecreasing, we observe from (2.6) that

$$\frac{u(x,y)}{m(x,y)} \le 1 + p\left(x, y\left(\int_{x_0}^{x} b(s,y)\, \frac{u(s,y)}{m(s,y)} ds\right)\right). \tag{2.7}$$

The inequality (2.7) may be treated as a one-dimensional Gronwall type inequality for any fixed y between y_0 to y, which implies that

$$u(x,y) \le F(x,y)\, m(x,y), \tag{2.8}$$

where $F(x,y)$ is as defined in (2.3). Now, from (2.5) and (2.8) we have

$$u(x,y) \le F(x,y)\left[a(x,y) + H(v(x,y))\right], \tag{2.9}$$

where

$$v(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(u(s,t))\, dsdt, \quad v(x,y_0) = v(x_0,y) = 0. \tag{2.10}$$

From (2.9) we have

$$G(u(x,y)) \le G(a(x,y)\, F(x,y)) + G(F(x,y))\, G(H(v(x,y))), \tag{2.11}$$

since G is subadditive and submultiplicative. Using (2.11) in (2.10) we see that the inequality

$$v(x,y) \le \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\left[G(a(s,t)\, F(s,t)) + G(F(s,t))\, G(H(v(s,t)))\right] dsdt,$$

is satisfied for all $(x,y) \in D$. Now fix $(\alpha,\beta) \in D_1$ such that $x_0 \le x \le \alpha \le x_1$, $y_0 \le y \le \beta \le y_1$ for $(x_1,y_1) \in D_1$, and set

$$A(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(a(s,t)\, F(s,t))\, dsdt,$$

then

$$v(x,y) \le A(\alpha,\beta) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(F(s,t))\, G(H(v(s,t)))\, dsdt, \tag{2.12}$$

for $x_0 \le x \le \alpha$, $y_0 \le y \le \beta$. Define

$$\begin{aligned} r(x,y) &= A(\alpha,\beta) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(F(s,t))\, G(H(v(s,t)))\, dsdt, \\ r(x,y_0) &= r(x_0,y) = A(\alpha,\beta), \end{aligned}$$

then

$$r_{xy}(x,y) = c(x,y)\, G(F(x,y))\, G(H(v(x,y))),$$

which, in view of (2.12), implies

$$r_{xy}(x,y) \le c(x,y)\, G(F(x,y))\, G(H(r(x,y))),$$

that is,

$$r_{xy}(x,y) \over G(H(r(x,y))) \le c(x,y)G(F(x,y)). \tag{2.13}$$

From (2.13), we observe that

$$\frac{G(H(r(x,y)))\,r_{xy}(x,y)}{G^2(H(r(x,y)))} \le c(x,y)G(F(x,y)) + \frac{r_x(x,y)\,G_y(H(r(x,y)))}{G^2(H(r(x,y)))}$$

that is,

$$\frac{\partial}{\partial y}\left(\frac{r_x(x,y)}{G(H(r(x,y)))}\right) \le c(x,y)G(F(x,y)).$$

By keeping x fixed in the above inequality, set $y = t$ and then integrating with respect to t from y_0 to β we have

$$\frac{r_x(x,\beta)}{G(H(r(x,\beta)))} \le \int_{y_0}^{\beta} c(x,t)G(F(x,t))\,dt. \tag{2.14}$$

From (2.4) and (2.14) we observe that

$$\Omega_x(r(x,\beta)) \le \int_{y_0}^{\beta} c(x,t)G(F(x,t))\,dt. \tag{2.15}$$

Now setting $x = s$ in (2.15) and then integrating with respect to s from x_0 to α, we have

$$\Omega(r(\alpha,\beta)) \le \Omega(r(x_0,\beta)) + \int_{x_0}^{\alpha}\int_{y_0}^{\beta} c(s,t)G(F(s,t))\,dsdt,$$

for $x_0 \le x \le \alpha$, $y_0 \le y \le \beta$. Since $(\alpha,\beta) \in D_1$ is arbitrary in $x_0 \le x \le \alpha \le x_1$ and $y_0 \le y \le \beta \le y_1$, we have

$$\begin{aligned} r(x,y) \quad &\le \quad \Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)G(a(s,t)F(s,t))\,dsdt\right)\right. \\ &\qquad \left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)G(F(s,t))\,dsdt\right], \end{aligned} \tag{2.16}$$

$x_0 \le x \le x_1$, $y_0 \le y \le y_1$. The desired bound in (2.2) follows from (2.9), (2.12) and (2.16). The subdomain D_1 of D is obvious. ∎

Another interesting and useful partial integral inequality is embodied in the following theorem.

Theorem 2. *Suppose (H_1), (H_3) – (H_5) are true. If*

$$\begin{aligned} u(x,y) \quad &\le \quad a(x,y) + q(x,y)\left(\int_{y_0}^{y} b(x,t)u(x,t)\,dt\right) \\ &\qquad + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)G(u(s,t))\,dsdt\right), \end{aligned} \tag{2.17}$$

for all $(x,y) \in D$*, then for all* $(x,y) \in D_2 \subset D$*,*

$$\begin{aligned} (2.18)\quad & u(x,y) \\ \leq\ & F_0(x,y)\left[a(x,y) + H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(a(s,t)\, F_0(s,t))\, dsdt\right)\right.\right.\right. \\ & \left.\left.\left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(F_0(s,t))\, dsdt\right]\right)\right], \end{aligned}$$

where

$$F_0(x,y) = 1 + q(x,y)\left(\int_{y_0}^{y} b(x,t)\exp\left(\int_{t}^{y} b(x,\eta)\, q(x,\eta)\, d\eta\right) dt\right), \tag{2.19}$$

Ω, Ω^{-1} *are as defined in Theorem 1, and*

$$\begin{aligned} & \Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(a(s,t)\, F_0(s,t))\, dsdt\right) \\ & + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(F_0(s,t))\, dsdt \in Dom\left(\Omega^{-1}\right), \end{aligned}$$

for all $(x,y) \in D_2$*.*

The details of the proof of this theorem follows by an argument similar to that in the proof of Theorem 1. We omit the details.

We next establish the following useful partial integral inequality which basically involves the comparison principle.

Theorem 3. *Suppose* (H_1) *and* (H_6) *are true; and let* $u(x,y)$*,* $b(x,y)$ *and* $p(x,y)$ *be as defined in* (H_2)*. If*

$$\begin{aligned} u(x,y) \leq\ & a(x,y) + p(x,y)\left(\int_{x_0}^{x} b(s,y)\, u(s,y)\, ds\right) \\ & + W\left(x, y, \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\, dsdt\right), \end{aligned} \tag{2.20}$$

for all $(x,y) \in D$*, then*

$$u(x,y) \leq F(x,y)\left[a(x,y) + W(x,y,r(x,y))\right], \tag{2.21}$$

for all $(x,y) \in D$*, where* $F(x,y)$ *is as defined in (2.3) and* $r(x,y)$ *is a solution of the integral equation*

$$r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,F(s,t)\left[a(s,t) + W(s,t,r(s,t))\right])\, dsdt, \tag{2.22}$$

existing on D*.*

Proof. Define

$$n(x,y) = a(x,y) + W\left(x, y, \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\, dsdt\right), \tag{2.23}$$

then (2.20) can be restated as

$$u(x,y) \leq n(x,y) + p(x,y)\left(\int_{x_0}^{x} b(s,y)\, u(s,y)\, ds\right). \tag{2.24}$$

Since $n(x,y)$ is positive, monotonic and nondecreasing, we observe from (2.24) that

$$\frac{u(x,y)}{n(x,y)} \le 1 + p(x,y)\left(\int_{x_0}^{x} b(s,y)\frac{u(s,y)}{n(s,y)}ds\right). \tag{2.25}$$

The inequality (2.25) may be treated as a one-dimensional Gronwall inequality for any fixed y between y_0 to y, which implies that

$$u(x,y) \le F(x,y)\, n(x,y), \tag{2.26}$$

where $F(x,y)$ is the function as defined in (2.3). Now, from (2.23) and (2.26), we have

$$u(x,y) \le F(x,y)\left[a(x,y) + W(x,y,z(x,y))\right], \tag{2.27}$$

where

$$z(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\,dsdt. \tag{2.28}$$

Using (2.27) in (2.28) we have

$$z(x,y) \le \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,F(s,t)\left[a(s,t) + W(s,t,z(s,t))\right])\,dsdt. \tag{2.29}$$

A suitable application of Theorem B given by Pelezar [17] to (2.22) and (2.29) yields

$$z(x,y) \le r(x,y), \tag{2.30}$$

where $r(x,y)$ is the solution of (2.22). Now, using (2.30) in (2.27) we obtain the desired bound in (2.21). ∎

Before leaving this section, we establish the following variant of Theorem 3 which can be used in some applications.

Theorem 4. *Suppose (H_1) and (H_6) are true; and let $u(x,y)$, $b(x,y)$ and $q(x,y)$ be as defined in (H_3). If*

$$\begin{aligned} u(x,y) \quad \le \quad & a(x,y) + q(x,y)\left(\int_{y_0}^{y} b(x,t)\,u(x,t)\,dt\right) \\ & + W\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\,dsdt\right), \end{aligned} \tag{2.31}$$

for all $(x,y) \in D$, then

$$u(x,y) \le F_0(x,y)\left[a(x,y) + W(x,y,r(x,y))\right], \tag{2.32}$$

for all $(x,y) \in D$, where $F_0(x,y)$ is as defined in (2.19) and $r(x,y)$ is a solution of the integral equation

$$r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,F_0(s,t)\left[a(s,t) + W(s,t,r(s,t))\right])\,dsdt, \tag{2.33}$$

existing on D.

The proof is omitted since it parallels that of the proof of Theorem 3.

3. Further Inequalities

In this sections we present a number of partial integral inequalities which can be used in several applications in the theory of nonlinear non-self-adjoint hyperbolic partial differential and integral equations. To establish some of our results in this section, we require the class of functions S as defined in [2].

A function $g:[0,\infty)\to[0,\infty)$ is said to belong to the class S if

(i) $g(u)$ is positive, nondecreasing and continuous for $u\geq 0$,
(ii) $\frac{1}{v}g(u)\leq g\left(\frac{u}{v}\right)$, $u>0$, $v\geq 1$.

In the following theorem we obtain the bound on the partial integral inequality involving two nonlinear terms on the right side.

Theorem 5. *Suppose (H_1), (H_4) and (H_5) are true; let $g\in S$, and $u(x,y)$, $b(x,y)$ and $c(x,y)$ be as defined in (H_2). If*

$$u(x,y) \leq a(x,y)+\int_{x_0}^{x} b(s,y)\,g(u(s,y))\,ds + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(u(s,t))\,dsdt\right), \tag{3.1}$$

for all $(x,y)\in D$, then for all $(x,y)\in D_3\subset D$,

$$\begin{aligned} &u(x,y) \\ \leq\ & U(x,y)\left[a(x,y)+H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,U(s,t))\,dsdt\right)\right.\right.\right. \\ &+\left.\left.\left.\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(U(s,t))\,dsdt\right]\right)\right], \end{aligned} \tag{3.2}$$

where

$$U(x,y) = E^{-1}\left[E(1)+\int_{x_0}^{x} b(s,y)\,ds\right], \tag{3.3}$$

$$E(r) = \int_{r_0}^{r}\frac{ds}{g(s)},\quad r\geq r_0>0, \tag{3.4}$$

E^{-1} is the inverse function of E; Ω, Ω^{-1} are as defined in Theorem 1, and

$$E(1)+\int_{x_0}^{x} b(s,y)\,ds\in Dom\left(E^{-1}\right),$$

and

$$\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,U(s,t))\,dsdt\right) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(U(s,t))\,dsdt\in Dom\left(\Omega^{-1}\right),$$

for all $(x,y)\in D_3$.

Proof. Define a function $m(x,y)$ as in (3.5), then (3.1) can be restated as

$$u(x,y)\leq m(x,y)+\int_{x_0}^{x} b(s,y)\,g(u(s,y))\,ds. \tag{3.5}$$

Since $m(x,y)$ is positive, monotonic, nondecreasing and $g \in S$, we observe from (3.5) that

$$\frac{u(x,y)}{m(x,y)} \leq 1 + \int_{x_0}^{x} b(s,y)\, g\left(\frac{u(s,y)}{m(s,y)}\right) ds. \tag{3.6}$$

The inequality (3.6) may be treated as a one-dimensional Bihari inequality (see [2]) for any fixed y between y_0 and y, which implies that

$$u(x,y) \leq U(x,y)\, m(x,y), \tag{3.7}$$

where $U(x,y)$ is as defined in (3.3). Now, by following the last argument as in the proof of Theorem 1, we obtain the desired bound in (3.2). ∎

We note that, if the inequality (3.1) in Theorem 5 is replaced by

$$\begin{aligned} u(x,y) \leq\ & a(x,y) + \int_{y_0}^{y} b(x,t)\, g(u(x,t))\, dt \\ & + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(u(s,t))\, ds dt\right), \end{aligned} \tag{3.8}$$

then the bound obtained in (3.2) is replaced by

$$\begin{aligned} & u(x,y) \\ \leq\ & U_0(x,y)\left[a(x,y) + H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(a(s,t)\, U_0(s,t))\, ds dt\right)\right.\right.\right. \\ & \left.\left.\left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(U_0(s,t))\, ds dt\right]\right)\right], \end{aligned} \tag{3.9}$$

where

$$U_0(x,y) = E^{-1}\left[E(1) + \int_{y_0}^{y} b(x,t)\, dt\right], \tag{3.10}$$

and E, E^{-1}; Ω, Ω^{-1}, are as defined in Theorem 5.

We next establish the following partial integral inequality which can be used in some applications.

Theorem 6. *Suppose* (H_1) *and* (H_6) *are true; let* $g \in S$ *and* $u(x,y)$ *and* $b(x,y)$ *be as defined in* (H_2). *If*

$$\begin{aligned} u(x,y) \leq\ & a(x,y) + \int_{x_0}^{x} b(s,y)\, g(u(s,y))\, ds \\ & + W\left(x, y, \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\, ds dt\right), \end{aligned} \tag{3.11}$$

for all $(x,y) \in D$, *then*

$$u(x,y) \leq U(x,y)\left[a(x,y) + W(x,y,r(x,y))\right], \tag{3.12}$$

for all $(x,y) \in D$, *where* $U(x,y)$ *is as defined in (3.3) and* $r(x,y)$ *is a solution of the integral equation*

$$r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,U(s,t)\left[a(s,t) + W(s,t,r(s,t))\right])\, ds dt, \tag{3.13}$$

existing on D.

The proof of this theorem follows by an argument similar to that given in the proof of Theorem 5 in view of the proof of Theorem 3. We omit the details.

We note that, if the inequality (3.11) in Theorem 6 is replaced by

$$u(x,y) \leq a(x,y)+\int_{y_0}^{y} b(x,t)\, g(u(x,t))\, dt + W\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\, dsdt\right), \tag{3.14}$$

then the bound obtained in (3.12) is replaced by

$$u(x,y) \leq U_0(x,y)\left[a(x,y)+W(x,y,r(x,y))\right], \tag{3.15}$$

where $U_0(x,y)$ is as defined in (3.10), and $r(x,y)$ is a solution of (3.13) when $U(x,y)$ is replaced by $U_0(x,y)$.

In concluding this section we note that:

(i) If the integral inequality (2.1) in Theorem 1 is replaced by

$$u(x,y) \leq a(x,y)+\int_{x_0}^{x} b(s,y)\left(u(s,y)+\int_{x_0}^{s} p(\xi,y)\, u(\xi,y)\, d\xi\right) ds + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(u(s,t))\, dsdt\right) \tag{3.16}$$

then by following a similar argument as in the proof of Theorem 1 and using the integral inequality established by Pachpatte [15, Theorem 1], we see that the bound obtained in (2.2) is replaced by

$$\begin{aligned} u(x,y) &\leq Z(x,y)\Bigg[a(x,y)+H\Bigg(\Omega^{-1}\Bigg[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(a(s,t)\, Z(s,t))\, dsdt\right) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(Z(s,t))\, dsdt\Bigg]\Bigg)\Bigg], \end{aligned} \tag{3.17}$$

where

$$Z(x,y) = 1+\int_{x_0}^{x} b(s,y)\exp\left(\int_{x_0}^{s}\left[b(\xi,y)+p(\xi,y)\right] d\xi\right) ds. \tag{3.18}$$

(ii) If the inequality (2.20) in Theorem 3 is replaced by

$$u(x,y) \leq a(x,y)+\int_{x_0}^{x} b(s,y)\left(u(s,y)+\int_{x_0}^{s} p(\xi,y)\, u(\xi,y)\, d\xi\right) ds + W\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\, dsdt\right), \tag{3.19}$$

then by following a similar argument as in the proof of Theorem 3 and using the integral inequality established by Pachpatte [15, Theorem 1], we see that the bound obtained in (2.21) is replaced by

$$u(x,y) \leq Z(x,y)\left[a(x,y)+W(x,y,r(x,y))\right], \tag{3.20}$$

where $Z(x,y)$ is as defined in (3.18) and $r(x,y)$ is the solution of (2.22) when $F(x,y)$ is replaced by $Z(x,y)$.

(iii) If the inequality (3.1) in Theorem 5 is replaced by

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x} b(s,y)\left(u(s,y) + \int_{x_0}^{s} b(\xi,y)\, g(u(\xi,y))\, d\xi\right) ds + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(u(s,t))\, dsdt\right), \tag{3.21}$$

then by following a similar argument as in the proof of Theorem 5 and using the integral inequality established by Pachpatte [16, Theorem 2], we see that the bound obtained in (3.2) is replaced by

$$\begin{aligned} & u(x,y) \\ \leq\ & Z_1(x,y)\left[a(x,y) + H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(a(s,t)\, Z_1(s,t))\, dsdt\right) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\, G(Z_1(s,t))\, dsdt\right]\right)\right], \end{aligned} \tag{3.22}$$

where

$$Z_1(x,y) = 1 + \int_{x_0}^{x} b(s,y)\, E^{-1}\left[E(1) + \int_{x_0}^{s} b(\xi,y)\, d\xi\right] ds, \tag{3.23}$$

$$E(r) = \int_{r_0}^{r} \frac{ds}{s + g(s)}, \quad r \geq r_0 > 0, \tag{3.24}$$

E^{-1} is the inverse of E, and Ω, Ω^{-1} are as defined in Theorem 1.

(iv) If the inequality (3.11) in Theorem 6 is replaced by

$$\begin{aligned} & u(x,y) \\ \leq\ & a(x,y) + \int_{x_0}^{x} b(s,y)\left(u(s,y) + \int_{x_0}^{s} b(\xi,y)\, g(u(\xi,y))\, d\xi\right) ds \\ & + W\left(x, y, \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\, dsdt\right), \end{aligned} \tag{3.25}$$

then by following a similar argument as in the proof of Theorem 6 and using the integral inequality established by Pachpatte [16, Theorem 2], we see that the bound obtained in (3.12) is replaced by

$$u(x,y) \leq Z_1(x,y)\left[a(x,y) + W(x,y,r(x,y))\right], \tag{3.26}$$

where $Z_1(x,y)$ is as defined in (3.23) and $r(x,y)$ is a solution of (3.13) when $U(x,y)$ is replaced by $Z_1(x,y)$.

Finally, we note that by replacing the first integrals on the right sides in (3.16) and (3.19) by

$$\int_{y_0}^{y} b(x,t)\left(u(x,t) + \int_{y_0}^{t} p(x,\eta)\, u(x,\eta)\, d\eta\right) dt,$$

and the first integrals on the right sides in (3.21) and (3.25) by

$$\int_{y_0}^{y} b(x,t)\left(u(x,t) + \int_{y_0}^{t} p(x,\eta)\, g(u(x,\eta))\, d\eta\right) dt,$$

we can very easily obtain bounds on the corresponding inequalities which may be convenient in some applications.

4. Use of the Riemann Function

In this section we establish some new and more general partial integral inequalities in two independent variables which can be used in some applications in the theory of nonlinear non-self-adjoint hyperbolic partial integrodifferential and integral equations of the more general type. The inequalities are established by solving the characteristic initial value problems using the Riemann method. Thus the Riemann functions associated with the hyperbolic partial differential equations appear in the bounds obtained on inequalities.

A useful and more general partial integral inequality is established in the following theorem.

Theorem 7. *Suppose (H_1), (H_4), (H_5), (H_7), (H_9) and (H_{10}) are true. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$M[v] = 0, \tag{4.1}$$

where M is the adjoint operator of the operator L defined by

$$L[\Psi] = \Psi_{st} + a_1\Psi_t + a_2\Psi, \tag{4.2}$$

in which $a_1 = -bp$, $a_2 = -[f + b(r+h)]$. Let $e(s,t;x,y)$ be the solution of the characteristic initial value problem

$$N[e] = 0, \tag{4.3}$$

where N is the adjoint operator of the operator T defined by

$$T[\Phi] = \Phi_{st} + b_1\Phi_t + b_2\Phi, \tag{4.4}$$

in which $b_1 = -bp$, $b_2 = -b(r-h)$. Let D^+ be a connected subdomain of D which contains P and on which $v \ge 0$ and $e \ge 0$. If $R \subset D^+$ and $u(x,y)$ satisfies

$$\begin{aligned} & u(x,y) \\ \le\; & a(x,y) + p(x,y)\left(\int_{x_0}^{x} b(s,y)\,u(s,y)\,ds\right) \\ & + r(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,u(s,t)\,dsdt\right) \\ & + h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,u(\xi,\eta)\,d\xi d\eta\right)dsdt\right) \\ & + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(u(s,t))\,dsdt\right), \end{aligned} \tag{4.5}$$

for all $(x,y) \in D$, then for $(x,y) \in D_3 \subset D$, $u(x,y)$ also satisfies

$$\begin{aligned} & u(x,y) \\ \le\; & F_1(x,y)\left[a(x,y) + H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t))\,F_1(s,t)\,dsdt\right)\right.\right.\right. \\ & \left.\left.\left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(F_1(s,t))\,dsdt\right]\right)\right], \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} & F_1(x,y) \\ =\; & f_0(x,y) + p(x,y)\left(\int_{x_0}^{x} b(s,y)\,f_0(s,y)\exp\left(\int_{s}^{x} b(\xi,y)\,p(\xi,y)\,d\xi\right)ds\right), \end{aligned} \tag{4.7}$$

in which

$$f_0(x,y) = 1 + r(x,y)\,Q(x,y) + h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\,Q(s,t)\,dsdt\right), \tag{4.8}$$

$$\begin{aligned} Q(x,y) &= \int_{x_0}^{x}\int_{y_0}^{y} e(s,t;x,y)\,b(s,t) \\ &\quad \cdot\left\{1 + h(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} v(\xi,\eta;s,t)\,b(\xi,\eta)\,d\xi d\eta\right)\right\} dsdt, \end{aligned} \tag{4.9}$$

and Ω, Ω^{-1} *are as defined in Theorem 1, and*

$$\begin{aligned} &\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,F_1(s,t))\,dsdt\right) \\ &+\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(F_1(s,t))\,dsdt \in Dom\left(\Omega^{-1}\right), \end{aligned}$$

for all $(x,y) \in D_3$.

Proof. Define a function $m(x,y)$ as in (2.5), then (4.5) can be restated as

$$\begin{aligned} &u(x,y) \\ \le\; & m(x,y) + p(x,y)\left(\int_{x_0}^{x} b(s,y)\,u(s,y)\,ds\right) \\ & + r(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,u(s,t)\,dsdt\right) \\ & + h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,u(\xi,\eta)\,d\xi d\eta\right) dsdt\right). \end{aligned} \tag{4.10}$$

Since $m(x,y)$ is positive, monotonic and nondecreasing, we observe from (4.10) that

$$\begin{aligned} &\frac{u(x,y)}{m(x,y)} \\ \le\; & 1 + p(x,y)\left(\int_{x_0}^{x} b(s,y)\,\frac{u(s,y)}{m(s,y)}ds\right) \\ & + r(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,\frac{u(s,t)}{m(s,t)}dsdt\right) \\ & + h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,\frac{u(\xi,\eta)}{m(\xi,\eta)}d\xi d\eta\right) dsdt\right). \end{aligned}$$

Now a suitable application of Theorem 7 given in [11] or Theorem 1 given in [14] yields

$$u(x,y) \le F_1(x,y)\,m(x,y), \tag{4.11}$$

where $F_1(x,y)$ is as defined in (4.7) in which $f_0(x,y)$ and $Q(x,y)$ are as given in (4.8) and (4.9). Using the definition of $m(x,y)$ and (4.11), we have

$$u(x,y) \le F_1(x,y)\left[a(x,y) + H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(u(s,t))\,dsdt\right)\right]. \tag{4.12}$$

Now by following a similar argument to that in the proof of Theorem 1, we obtain the desired bound in (4.6). The subdomain D_3 of D is obvious. ∎

We next establish the following variant of Theorem 7 which can be used in certain applications.

Theorem 8. *Suppose* (H_1), (H_4), (H_5), (H_8) - (H_{10}) *are true. Let* $v(s,t;x,y)$ *be the solution of the characteristic initial value problem*

$$M[v]=0, \tag{4.13}$$

where M *is the adjoint operator of the operator* L *defined by*

$$L[\Psi]=\Psi_{st}+a_1\Psi_s+a_2\Psi, \tag{4.14}$$

in which $a_1=-bq$, $a_2=-[f+b(r+h)]$. *Let* $e(s,t;x,y)$ *be the solution of the characteristic initial value problem*

$$N[e]=0, \tag{4.15}$$

where N *is the adjoint operator of the operator* T *defined by*

$$T[\Phi]=\Phi_{st}+b_1\Phi_s+b_2\Phi, \tag{4.16}$$

in which $b_1=-bq$, $b_2=-b(r-h)$. *Let* D^+ *be a connected subdomain of* D *which contains* P *and on which* $v\geq 0$ *and* $e\geq 0$. *If* $R\subset D^+$ *and* $u(x,y)$ *satisfies*

$$\begin{aligned} &u(x,y)\\ \leq\ & a(x,y)+q(x,y)\left(\int_{y_0}^{y} b(x,t)\,u(x,t)\,dt\right)\\ &+r(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,u(s,t)\,dsdt\right)\\ &+h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,u(\xi,\eta)\,d\xi d\eta\right)dsdt\right)\\ &+H\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(u(s,t))\,dsdt\right), \end{aligned} \tag{4.17}$$

for all $(x,y)\in D$ *then, for* $(x,y)\in D_4\subset D$, $u(x,y)$ *also satisfies*

$$\begin{aligned} &u(x,y)\\ \leq\ & F_2(x,y)\left[a(x,y)+H\left(\Omega^{-1}\left[\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,F_2(s,t))\,dsdt\right)\right.\right.\right.\\ &\left.\left.\left.+\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(F_2(s,t))\,dsdt\right]\right)\right], \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} &F_2(x,y)\\ =\ & f_0(x,y)+q(x,y)\left(\int_{y_0}^{y} b(x,t)\,f_0(x,t)\exp\left(\int_{t}^{y} b(x,\eta)\,q(x,\eta)\,d\eta\right)dt\right), \end{aligned} \tag{4.19}$$

in which $f_0(x,y)$ *and* $Q(x,y)$ *are as defined in Theorem 7; and* Ω, Ω^{-1} *are as defined in Theorem 1 and*

$$\begin{aligned} &\Omega\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(a(s,t)\,F_2(s,t))\,dsdt\right)\\ &+\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,G(F_2(s,t))\,dsdt\in Dom\left(\Omega^{-1}\right), \end{aligned}$$

for all $(x,y)\in D_3$.

The proof of this theorem follows by an argument similar to that in the proof of Theorem 7 with suitable modifications. We omit the details.

We next establish the following partial integral inequalities which can be used in certain applications in the theory of nonlinear non-self-adjoint hyperbolic partial integrodifferential and integral equations of the more general type.

Theorem 9. *Suppose* (H_1), (H_6), (H_9) *and* (H_{10}) *are true; and let* $u(x,y)$, $b(x,y)$, $p(x,y)$, $h(x,y)$ *and* $f(x,y)$ *be as defined in* (H_7). *Let* $v(s,t;x,y)$ *be the solution of the characteristic initial value problem (4.1) in which* M *is the adjoint operator of the operator* L *defined by (4.2); and let* $e(s,t;x,y)$ *be the solution of the characteristic initial value problem (4.3) in which* N *is the adjoint operator of the operator* T *defined by (4.4). Let* D^+ *be a connected subdomain of* D *which contains* P *and on which* $v\geq 0$ *and* $e\geq 0$. *If* $R\subset D^+$ *and* $u(x,y)$ *satisfies*

$$\begin{aligned} (4.20)\qquad & u(x,y) \\ \leq\ & a(x,y)+p(x,y)\left(\int_{x_0}^{x} b(s,y)\,u(s,y)\,ds\right) \\ & +r(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,u(s,t)\,dsdt\right) \\ & +h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,u(\xi,\eta)\,d\xi d\eta\right)dsdt\right) \\ & +W\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\,dsdt\right), \end{aligned}$$

for all $(x,y)\in D$, *then*

$$u(x,y)\leq F_1(x,y)\left[a(x,y)+W(x,y,r(x,y))\right], \tag{4.21}$$

for all $(x,y)\in D$, *where* $F_1(x,y)$ *is as defined in Theorem 5 and* $r(x,y)$ *is a solution of the integral equation*

$$r(x,y)=\int_{x_0}^{x}\int_{y_0}^{y} K\left(x,y,s,t,F_1(s,t)\left[a(s,t)+W(s,t,r(s,t))\right]\right)dsdt, \tag{4.22}$$

existing on D.

Theorem 10. *Suppose* (H_1), (H_6), (H_9) *and* (H_{10}) *are true; and let* $u(x,y)$, $b(x,y)$, $q(x,y)$, $r(x,y)$, $h(x,y)$ *and* $f(x,y)$ *be as defined in* (H_8). *Let* $v(s,t;x,y)$ *be the solution of the characteristic initial value problem (4.13) in which* M *is the adjoint operator of the operator* L *defined by (4.14); and let* $e(s,t;x,y)$ *be the solution of the characteristic initial value problem (4.15) in which* N *is the adjoint operator of the operator* T *defined by (4.16). Let* D^+ *be a connected subdomain of*

D which contains P and on which $v \geq 0$ and $e \geq 0$. If $R \subset D^{+}$ and $u(x,y)$ satisfies

$$\begin{aligned} (4.23) \qquad & u(x,y) \\ \leq\ & a(x,y) + q(x,y)\left(\int_{y_0}^{y} b(x,t)\,u(x,t)\,dt\right) \\ & + r(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,u(s,t)\,dsdt\right) \\ & + h(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,u(\xi,\eta)\,d\xi d\eta\right)dsdt\right) \\ & + W\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,u(s,t))\,dsdt\right), \end{aligned}$$

for all $(x,y) \in D$, then

$$u(x,y) \leq F_2(x,y)\left[a(x,y) + W(x,y,r(x,y))\right], \tag{4.24}$$

for all $(x,y) \in D$, where $F_2(x,y)$ is as defined in Theorem 6 and $r(x,y)$ is a solution of the integral equation

$$r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K\left(x,y,s,t,F_2(s,t)\left[a(s,t) + W(s,t,r(s,t))\right]\right)dsdt, \tag{4.25}$$

existing on D.

The details of the proofs of Theorems 9 and 10 follow by using similar arguments to those in the proofs of Theorems 7 and 8, by making use of the arguments of Theorems 3 and 4, and we leave the details to the reader.

5. Some Applications

In this section we present some applications of our results to the boundedness, uniqueness and behavioural relationships of the solutions of some nonlinear non-self-adjoint hyperbolic partial differential and integrodifferential equations. There appear to be many applications of the inequalities established in this paper but those presented here are sufficient to convey the importance of our results to the literature. These applications are not stated as theorems so as not to obscure the main ideas with technical details.

Example 1. *As a first application, we discuss the boundedness of the solution of a nonlinear non-self-adjoint hyperbolic partial differential equation*

$$u_{xy}(x,y) = \{b_0(x,y)\,u(x,y)\}_y + A(x,y,u(x,y)) + f_1(x,y) \tag{5.1}$$

with the boundary conditions prescribed on $x = x_0$ and $y = y_0$, where all functions are defined and continuous on their respective domains of definitions and such that

$$|b_0(x,y)| \leq b(x,y), \tag{5.2}$$

$$|A(x,y,u)| \leq K(x,y,|u|), \tag{5.3}$$

where $b(x,y)$ and $K(x,y,s,t,\Phi) = k(s,t,\Phi)$ are as defined in Theorem 3. Let the boundary conditions be such that the given equation (5.1) is equivalent to the integral equation

$$u(x,y) = a_0(x,y) + \int_{x_0}^{x} b_0(s,y)\,u(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} A(s,t,u(s,t))\,dsdt, \tag{5.4}$$

where $a_0(x,y)$ *is computed from* $f_1(x,y)$ *and the given boundary conditions. We assume that*

$$|a_0(x,y)| \leq a(x,y), \tag{5.5}$$

where $a(x,y)$ *is as defined in Theorem 3. Using (5.2), (5.3) and (5.5) in (5.4), we have*

$$|u(x,y)| \leq a(x,y) + \int_{x_0}^{x} b(s,y)|u(s,y)|\,ds + \int_{x_0}^{x}\int_{y_0}^{y} K(s,t,|u(s,t)|)\,dsdt.$$

Now, a suitable application of Theorem 3 with $p(x,y) = 1$, $W(x,y,\Phi) = \Phi$ *and* $K(x,y,s,t,\Phi) = K(s,t,\Phi)$ *yields*

$$|u(x,y)| \leq F^*(x,y)[a(x,y) + r(x,y)], \tag{5.6}$$

where $F^*(x,y)$ *is obtained by substituting* $p(x,y) = 1$ *in (2.3) and* $r(x,y)$ *is a solution of the integral equation*

$$r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(s,t,F^*(s,t)[a(s,t) + r(s,t)])\,dsdt, \tag{5.7}$$

existing on D. *If the right hand side of (5.6) is bounded, then we obtain the boundedness of the solution* $u(x,y)$ *of (5.1).*

Example 2. *As a second application, we discuss the uniqueness of solutions of equation (5.1). We assume that the functions involved in (5.1) satisfy*

$$|b_0(x,y)| \leq b(x,y), \tag{5.8}$$

$$|A(x,y,u) - A(x,y,\bar{u})| \leq K(x,y,|u-\bar{u}|), \tag{5.9}$$

where $b(x,y)$ *and* $K(x,y,\Phi)$ *are as defined in Example 1. Let the boundary conditions be such that the given equation is equivalent to the integral equation (5.4). Then for any two solutions* $u = u(x,y)$ *and* $\bar{u} = \bar{u}(x,y)$ *of (5.1) with the given boundary conditions we have*

$$\begin{aligned} u - \bar{u} &= a_0(x,y) - \bar{a}_0(x,y) + \int_{x_0}^{x} b_0(s,y)\{u-\bar{u}\}\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y}\{A(s,t,u) - A(s,t,\bar{u})\}\,dsdt, \end{aligned} \tag{5.10}$$

where $a_0(x,y)$ *and* $\bar{a}_0(x,y)$ *depend on the given boundary conditions and the function* $f_1(x,y)$. *Using (5.8) and (5.9) in (5.10) and further assuming that* $|a_0(x,y) - \bar{a}_0(x,y)| \leq \varepsilon$, *for arbitrary* $\varepsilon > 0$, *we have*

$$|u-\bar{u}| \leq \varepsilon + \int_{x_0}^{x} b(s,y)|u-\bar{u}|\,ds + \int_{x_0}^{x}\int_{y_0}^{y} K(s,t,|u-\bar{u}|)\,dsdt.$$

Now, a suitable application of Theorem 3 with $p(x,y) = 1$, $W(x,y,\Phi) = \Phi$ *and* $K(x,y,s,t,\Phi) = K(s,t,\Phi)$ *and* $a(x,y) = \varepsilon$ *gives*

$$|u-\bar{u}| \leq F^*(x,y)[\varepsilon + r(x,y)], \tag{5.11}$$

where $F^*(x,y)$ *is as defined in Example 1 and* $r(x,y)$ *is the solution of the integral equation (5.7) when* $a(x,y)$ *is replaced by* ε. *If equation (5.7) with* $a(x,y) = \varepsilon$ *admits only an identically zero solution, then from (5.11) we observe that* $u = \bar{u}$, *since* $\varepsilon > 0$ *is arbitrary, and hence there is at most one solution of the equation (5.1).*

Example 3. *Our third application is an example of behavioural relationships between the solutions of nonlinear non-self-adjoint hyperbolic partial integrodifferential equation*

$$
\begin{aligned}
(5.12) \qquad & u_{xy}(x,y) \\
= \; & \{b_0(x,y)\,u(x,y)\}_y + A(x,y,u(x,y)) + f_1(x,y) \\
& + H_1\left[x,y,\int_{x_0}^{x}\int_{y_0}^{y} K_1(x,y,s,t,u(s,t))\,dsdt\right],
\end{aligned}
$$

with the boundary conditions prescribed on $x = x_0$ and $y = y_0$ and the nonlinear non-self-adjoint hyperbolic partial integrodifferential equation

$$
\begin{aligned}
(5.13) \qquad & z_{xy}(x,y) \\
= \; & \{b_0(x,y)\,z(x,y)\}_y + A_0(x,y,z(x,y)) + g_1(x,y) \\
& + H_0\left[x,y,\int_{x_0}^{x}\int_{y_0}^{y} K_0(x,y,s,t,z(s,t))\,dsdt\right] + B(x,y,z(x,y))
\end{aligned}
$$

with the boundary conditions prescribed on $x = x_0$ and $y = y_0$, where all the functions in (5.12) and (5.13) are defined and continuous on their respective domains of definition and are such that

$$
\begin{aligned}
(5.14) \qquad |b_0(x,y)| &\leq b(x,y), \\
(5.15) \qquad |A_0(x,y,z) - A(x,y,u)| &\leq b(x,y)\,|z-u|, \\
(5.16) \qquad |B(x,y,z)| &\leq K(x,y,|z|), \\
(5.17) \qquad |K_0(x,y,s,t,z) - K_1(x,y,s,t,u)| &\leq b(s,t)\,|z-u|, \\
(5.18) \qquad |H_0[x,y,\bar{z}] - H_1[x,y,\bar{u}]| &\leq f(x,y)\,|\bar{z}-\bar{u}|,
\end{aligned}
$$

where $b(x,y)$, $f(x,y)$ and $K(x,y,s,t,\Phi) = K(s,t,\Phi)$ are as defined in Theorem 9. The equations (5.12) and (5.13) are equivalent to the integral equations

$$
\begin{aligned}
(5.19) \qquad & u(x,y) \\
= \; & a_0(x,y) + \int_{x_0}^{x} b_0(s,y)\,u(s,y)\,dy + \int_{x_0}^{x}\int_{y_0}^{y} A(s,t,u(s,t))\,dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} H_1\left[s,t,\int_{x_0}^{s}\int_{y_0}^{t} K_1(s,t,\xi,\eta,u(\xi,\eta))\,d\xi d\eta\right] dsdt
\end{aligned}
$$

and

$$
\begin{aligned}
(5.20) \qquad & z(x,y) \\
= \; & \bar{a}_0(x,y) + \int_{x_0}^{x} b_0(s,y)\,z(s,y)\,dy + \int_{x_0}^{x}\int_{y_0}^{y} A_0(s,t,z(s,t))\,dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} H_0\left[s,t,\int_{x_0}^{s}\int_{y_0}^{t} K_0(s,t,\xi,\eta,z(\xi,\eta))\,d\xi d\eta\right] dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} B(s,t,z(s,t))\,dsdt,
\end{aligned}
$$

where $a_0(x,y)$ and $\bar{a}_0(x,y)$ depend on the given boundary conditions and $f_1(x,y)$ and $g_1(x,y)$ respectively. We assume that

$$
(5.21) \qquad |\bar{a}_0(x,y) - a_0(x,y)| \leq a(x,y),
$$

where $a(x,y)$ is as defined in Theorem 9. From (5.19) and (5.20) we have

$$
\begin{aligned}
(5.22) \qquad & z(x,y) - u(x,y) \\
= \; & \bar{a}_0(x,y) - a_0(x,y) + \int_{x_0}^{x} b_0(s,y)\{z(s,y) - u(s,y)\}\,ds \\
& + \int_{x_0}^{x}\int_{y_0}^{y} \{A_0(s,t,z(s,t)) - A(s,t,u(s,t))\}\,dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} \left\{ H_0\left[s,t,\int_{x_0}^{s}\int_{y_0}^{t} K_0(s,t,\xi,\eta,z(\xi,\eta))\,d\xi d\eta\right] \right. \\
& \left. - H_1\left[s,t,\int_{x_0}^{s}\int_{y_0}^{t} K_1(s,t,\xi,\eta,u(\xi,\eta))\,d\xi d\eta\right]\right\} dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} B(s,t,z(s,t))\,dsdt.
\end{aligned}
$$

Using (5.14) - (5.18), (5.21) and $|z| - |u| \leq |z-u|$ in (5.22) and assuming that the solution $u(x,y)$ of (5.12) is bounded by M_0, where $M_0 > 0$ is a constant, we have

$$
\begin{aligned}
& |z(x,y) - u(x,y)| \\
\leq \; & a(x,y) + \int_{x_0}^{x} b(s,y)\,|z(s,y) - u(s,y)|\,ds \\
& + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\,|z(s,t) - u(s,t)|\,dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)\,|z(\xi,\eta) - u(\xi,\eta)|\,d\xi d\eta\right) dsdt \\
& + \int_{x_0}^{x}\int_{y_0}^{y} K(s,t,M_0 + |z(s,t) - u(s,t)|)\,dsdt.
\end{aligned}
$$

Now a suitable application of Theorem 9 with $p(x,y) = r(x,y) = h(x,y) = 1$ and $K(x,y,s,t,\Phi) = K(s,t,\Phi)$ and $W(x,y,\Phi) = \Phi$ yields

$$
(5.23) \qquad |z(x,y) - u(x,y)| \leq F_1^*(x,y)\,[a(x,y) + r(x,y)],
$$

where $F_1^*(x,y)$ is obtained by substituting $p(x,y) = r(x,y) = h(x,y) = 1$ in (4.7) and $r(x,y)$ is a solution of the integral equation

$$
(5.24) \qquad r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(s,t,M_0 + F_1^*(s,t)\,[a(s,t) + r(s,t)])\,dsdt.
$$

If the right hand side in (5.23) is bounded, then we obtain the relative boundedness of the solutions $z(x,y)$ and $u(x,y)$ of (5.13) and (5.12). If $a(x,y)$ in (5.23) is small enough and say less than ε, where $\varepsilon > 0$ is arbitrary, and if the equation (5.24) admits only an identically zero solution, and if $F^*(x,y)$ in (5.22) is bounded and $\varepsilon \to 0$, then we obtain $|z(x,y) - u(x,y)| \to 0$, which gives the equivalence between the solutions of (5.12) and (5.13).

We note that Theorems 3 and 8 can be used to study the continuous dependence of the solutions of (5.1) and (5.13) by following similar arguments to those in [11] with suitable modifications. We also note that the inequalities established in Theorems 4 and 10 can be used to establish similar results as given in Examples

1 - 3 for the corresponding nonlinear non-self-adjoint hyperbolic partial differential and integrodifferential equations of the forms

$$u_{xy}(x,y) = \{b_0(x,y)\,u(x,y)\}_x + A(x,y,u(x,y)) + f_1(x,y), \tag{5.25}$$

$$\begin{aligned} u_{xy}(x,y) &= \{b_0(x,y)\,u(x,y)\}_x + A(x,y,u(x,y)) + f_1(x,y) \\ &\quad + H_1\left[x,y,\int_{x_0}^{x}\int_{y_0}^{y} K_1(x,y,s,t,u(s,t))\,dsdt\right], \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} &z_{xy}(x,y) \\ &= \{b_0(x,y)\,z(x,y)\}_x + A_0(x,y,z(x,y)) + g_1(x,y) \\ &\quad + H_0\left[x,y,\int_{x_0}^{x}\int_{y_0}^{y} K_0(x,y,s,t,z(s,t))\,dsdt\right] + B(x,y,z(x,y)), \end{aligned} \tag{5.27}$$

with the boundary conditions prescribed on $x = x_0$ and $y = y_0$ and under some suitable conditions on the functions involved therein.

In concluding this paper, we note that the integral inequalities established in Theorems 1 - 4 can be used to study the boundedness, uniqueness and continuous dependence of the solutions of nonlinear non-self-adjoint Volterra integral equations of the type

$$\begin{aligned} u(x,y) &= f(x,y) + \int_{x_0}^{x} K_1[x,y,s,u(s,y)]\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} K_2[x,y,s,t,u(s,t)]\,dsdt \end{aligned} \tag{5.28}$$

and

$$\begin{aligned} u(x,y) &= f(x,y) + \int_{y_0}^{y} K_1[x,y,t,u(x,t)]\,dt \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} K_2[x,y,s,t,u(s,t)]\,dsdt, \end{aligned} \tag{5.29}$$

under some suitable conditions on the functions involved in (5.28) and (5.29). We also note that the integral inequalities established in Theorems 7 - 10 can be used to study the boundedness, uniqueness and continuous dependence of the solutions of nonlinear non-self-adjoint Volterra integral equations of the more general type

$$\begin{aligned} &u(x,y) \\ &= f(x,y) + \int_{x_0}^{x} K_1[x,y,s,u(s,y)]\,ds + \int_{x_0}^{x}\int_{y_0}^{y} K_2[x,y,s,t,u(s,t)]\,dsdt \\ &\quad \int_{x_0}^{x}\int_{y_0}^{y} K_3\left[x,y,s,t,\int_{x_0}^{s}\int_{y_0}^{t} K_4(s,t,\xi,\eta,u(\xi,\eta))\,d\xi d\eta\right] dsdt \\ &\quad + W_0\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K_0(x,y,s,t,u(s,t))\,dsdt\right), \end{aligned} \tag{5.30}$$

and

$$
\begin{aligned}
(5.31) \qquad & u(x,y) \\
= \; & f(x,y) + \int_{y_0}^{y} K_1 [x,y,t,u(x,t)]\, dt + \int_{x_0}^{x}\int_{y_0}^{y} K_2 [x,y,s,t,u(s,t)]\, dsdt \\
& \int_{x_0}^{x}\int_{y_0}^{y} K_3 \left[x,y,s,t,\int_{x_0}^{s}\int_{y_0}^{t} K_4(s,t,\xi,\eta,u(\xi,\eta))\, d\xi d\eta\right] dsdt \\
& + W_0\left(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K_0(x,y,s,t,u(s,t))\, dsdt\right),
\end{aligned}
$$

under some suitable conditions on the functions involved in (5.30) and (5.31). Other applications of some of the inequalities established in this paper will appear elsewhere.

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