On the uniqueness of solutions to a class of singular anisotropic elliptic boundary value problems

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Abstract. In this paper we are concerned with the uniqueness results for the following singular anisotropic problem

$$\sum_{i=1}^{N-1} f_i(u)u_{x_i x_i} + u_{yy} + p(x)g(u) = 0 \quad \text{in } \Omega$$

with zero boundary conditions, where Ω is a bounded domain in $\mathbf{R}^N$, p is positive continuous function on $\overline{\Omega}$, $N \geq 2$ and g is a positive nonincreasing C^1-function on $(0, \infty)$. Under various assumptions on f_i, we prove that if there exists a solution $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ satisfying an a-priori bound on the second derivatives from either below (Theorem 1) or from above (Theorem 2), then this solution is the only one. Our paper extends previous results of Lair and Shaker [6] to more than two independent variables and to more general nonlinearities.

Key words: anisotropic problem, uniqueness result, maximum principle

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1 Introduction and the main results

Singular anisotropic boundary value problems arise naturally when studying many concrete situations. We refer to Čanić-Keyfitz [1] for the study of self-similar solutions of conservation laws in two dimensions. We also mention Ding-Liu [5] where it is studied another anisotropic problem in the plane. Their model is closely related to the phase transition problem in anisotropic superconductivity with "thermal noise" term.

In [2], Choi, Lazer and McKenna studied a problem that is linked to an equation arising in fluid dynamics. They proved that the singular elliptic boundary value problem

$$\begin{cases} u^a u_{xx} + u^b u_{yy} + p(x,y) = 0, & (x,y) \in \Omega \\ u = 0, & (x,y) \in \partial\Omega \end{cases} \tag{1}$$

has a positive classical solution, where $\Omega \subset \mathbf{R}^2$ is a bounded convex domain with smooth boundary, p is a positive Hölder continuous function and the constants a, b satisfy $a > b \geq 0$. Choi, Lazer and

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McKenna also developed a new comparison principle for quasilinear problems that is based on the method of sub- and super-solutions.

Recently Choi and McKenna [3] removed the assumption that the dimension be restricted to two, but they also retained the convexity assumption which is crucial in the construction of a super-solution ψ, satisfying the boundary conditions. More exactly, they showed that the boundary value problem

$$\begin{cases} \sum_{i=1}^{N} u^{a_i} u_{x_i x_i} + p(x) = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2}$$

has at least one positive classical solution u, such that $u(x) \le \psi(x)$ for all $x \in \Omega$, where $\Omega \subset \mathbf{R}^N$ $(N \ge 1)$ is a bounded convex domain with smooth boundary and $a_1 \ge a_2 \ge \cdots \ge a_N \ge 0$, with $a_1 > a_N$. Choi and McKenna point out that the most significant omission of their paper is the absence of any information on the uniqueness of solutions. In this direction there are known very few results which hold only for the two dimensional case.

Lair and Shaker proved in [6] a uniqueness result related to (1) and they required neither the domain Ω to be convex nor the function p to be as smooth as in [2]. They made only the assumption that there is some solution u for which u_{xx} is bounded above appropriately. In their paper there are distinguished two different situations: $a - b \ge 1$, resp., $a - b < 1$.

Reichel [7] established that problem (1) has at most one positive classical solution. It is assumed that

$$p(\tau_1 x, \tau_2 y) \ge p(x, y) \quad \text{for all } (x, y) \in \Omega,\ \tau_i \in [0, 1]$$

and the bounded domain Ω (with $0 \in \Omega$) satisfies an interior rectangle condition, i.e., for each $(x, y) \in \partial\Omega$ the rectangle $\{(\tau_1 x, \tau_2 y) :\ \tau_i \in [0, 1)\}$ is a subset of Ω.

It is natural to ask us if it is possible to give a uniqueness result which holds for more general degenerate quasilinear operators and for a larger class of functions p, with no assumption on the geometry of the domain or the dimension of the space.

For this aim, we consider the singular anisotropic elliptic boundary value problem

$$\begin{cases} \sum_{i=1}^{N-1} f_i(u)\, u_{x_i x_i} + u_{yy} + p(x)\, g(u) = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N \ge 2$ and p is a positive continuous function on $\overline{\Omega}$. We have denoted the last coordinate x_N by y and we shall use notation x' for the first $(N-1)$ coordinates.

Throughout this paper, we assume that the following hypotheses are fulfilled

$(\mathbf{H}_1)$ $f_i, g : (0, \infty) \to (0, \infty)$, $i = \overline{1, N-1}$ are C^1-functions;

$(\mathbf{H}_2)$ f_i, $i = \overline{1, N-1}$ is nondecreasing on $(0, \infty)$ and g is nonincreasing on $(0, \infty)$.

Since Ω is bounded, we can make a translation of the domain so that it lies in the interior of the strip $\mathbf{R}^{N-1} \times [0, \ell]$ for some $\ell > 0$. The fact that $p \in C(\overline{\Omega})$ is a positive function implies the existence of α, $\beta > 0$ such that $p(x) \in [\alpha, \beta]$ for each $x \in \overline{\Omega}$.

Set

$$D = \{y \in [0,\ell] : \ \exists x' \text{ such that } (x',y) \in \overline{\Omega}\}.$$

We can suppose, without loss of generality, that $\ell \notin D$.

Let ψ be the unique positive function defined by

$$\int_0^{\psi(y)} \frac{1}{g(t)}\, dt = \frac{\beta}{2}(\ell y - y^2), \quad \text{for any } y \in [0,\ell]. \tag{4}$$

It is obvious that

$$\max_{y\in D} \psi(y) \le \max_{y\in[0,\ell]} \psi(y) = A, \tag{5}$$

where $A > 0$ is uniquely defined by

$$\int_0^{A} \frac{1}{g(t)}\, dt = \frac{\beta}{8}\ell^2. \tag{6}$$

We also assume

($\mathbf{H}_3$) $\quad f_1' > 0$ on $(0, A]$.

In the first result of this paper we impose the condition

($\mathbf{C}_1$) $\quad$ there exists and is finite $\lim\limits_{x\searrow 0} \dfrac{f_1 f_i'}{f_1'}(x)$, for all $i = \overline{2, N-1}$.

In view of this hypothesis we observe that for any $i = \overline{2, N-1}$ it makes sense to define

$$m_i = \min_{[0,A]} \frac{\left(\frac{f_i}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'} = \min_{[0,A]} \left(f_i - \frac{f_i' f_1}{f_1'}\right) \quad \text{and} \quad M_i = \max_{[0,A]} \frac{\left(\frac{f_i}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'} = \max_{[0,A]} \left(f_i - \frac{f_i' f_1}{f_1'}\right).$$

For any $x \in \Omega$ we define the sets

$$P_x = \{2 \le i \le N-1;\ u_{x_i x_i}(x) \ge 0\} \quad \text{and} \quad N_x = \{2 \le i \le N-1;\ u_{x_i x_i}(x) < 0\}.$$

Our first result asserts that the existence of a positive solution $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ of (3) ensures its uniqueness, provided that the expression $\sum\limits_{i\in P_x} m_i u_{x_i x_i} + \sum\limits_{i\in N_x} M_i u_{x_i x_i} + u_{yy}$ is bounded below appropriately.

Theorem 1 *Assume* ($\mathbf{H}_1$)-($\mathbf{H}_3$) *and* ($\mathbf{C}_1$) *hold. There exists a positive constant K_1, depending on f_1, g, p and Ω, such that if u is a positive solution of (3) satisfying*

$$\sum_{i\in P_x} m_i u_{x_i x_i} + \sum_{i\in N_x} M_i u_{x_i x_i} + u_{yy} > -K_1 \quad \text{in } \Omega \tag{7}$$

then u is the unique solution of (3).

We now drop the assumption ($\mathbf{C}_1$) but we require

($\mathbf{C}_2$) $\quad \dfrac{f_i}{f_1}$, $i = \overline{2, N-1}$ is nonincreasing on $(0, \infty)$.

Our next theorem shows that the uniqueness of solution to (3) is assured if we find a positive solution $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ with the property that $u_{x_1x_1} + \sum_{i\in P_x} \frac{f_i(u)}{f_1(u)} u_{x_ix_i} + \sum_{i\in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_ix_i}$ is bounded above appropriately.

Theorem 2 *Assume* $(\mathbf{H}_1)$*-*$(\mathbf{H}_3)$ *and* $(\mathbf{C}_2)$ *hold. There exists a non-negative constant* K_2 *depending on* f_1*,* g*,* p *and* Ω*, such that if* u *is a positive solution of problem (3) satisfying*

$$u_{x_1x_1} + \sum_{i\in P_x} \frac{f_i(u)}{f_1(u)} u_{x_ix_i} + \sum_{i\in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_ix_i} < K_2 \quad \text{in } \Omega \tag{8}$$

then u *is the unique solution of (3).*

2 An auxiliary result

In this section we prove that the number A given by (6) is an upper bound for every positive classical solution of problem (3). To this end, we make use of a comparison lemma on a class of quasilinear elliptic equations established in Choi-McKenna [3]. In view of this result we can obtain L^∞ bounds on the solutions to this class of equations using the method of sub- and super-solutions. Consider the problem

$$\begin{cases} \displaystyle\sum_{i=1}^{N-1} f_i(x,u)\, u_{x_ix_i} + u_{yy} + p(x)\, g(x,u) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega, \end{cases} \tag{9}$$

with $u_0|_{\partial\Omega} \geq 0$, where the functions f_i, g and p satisfy the assumptions

$(\mathbf{A}_1)$ $f_i : \Omega \times [0,\infty) \to [0,\infty)$ is continuous and $f_i(x,\cdot)$ is nondecreasing for each $x \in \Omega$;

$(\mathbf{A}_2)$ $g : \Omega \times (0,\infty) \to (0,\infty)$ is continuous, and $g(x,\cdot)$ is nonincreasing for each $x \in \Omega$;

$(\mathbf{A}_3)$ $p : \overline{\Omega} \to \mathbf{R}$ is continuous, and there exist positive constants α and β such that

$$0 < \alpha \leq p(x) \leq \beta \quad \text{for all } x \in \overline{\Omega}.$$

Assume that

$(\mathbf{L})$ There exists a sub-solution $\varphi \in C(\overline{\Omega}) \cap C^2(\Omega)$ with $\varphi > 0$ on Ω satisfying

$$\sum_{i=1}^{N-1} f_i(x,\varphi)\, \varphi_{x_ix_i} + \varphi_{yy} + p(x)\, g(x,\varphi) > 0, \quad \text{in } \Omega$$

$$\varphi_{x_ix_i} \leq 0, \quad \text{in } \Omega, \text{ for any } i = 1,2,\cdots,N-1,$$

and $\varphi \leq u_0$ on $\partial\Omega$.

$(\mathbf{U})$ There exists a super-solution $\psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ with $\psi > 0$ in Ω satisfying

$$\sum_{i=1}^{N-1} f_i(x,\psi)\, \psi_{x_ix_i} + \psi_{yy} + p(x)\, g(x,\psi) \leq 0, \quad \text{in } \Omega$$

$$\psi_{x_ix_i} \leq 0, \quad \text{in } \Omega, \text{ for any } i = 1,2,\cdots,N-1,$$

and $\psi \geq u_0$ on $\partial\Omega$.

Lemma 1 *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$, $(\mathbf{L})$ *and* $(\mathbf{U})$ *hold. Then any positive solution* u *of (3) satisfies* $u \le A$ *in* $\overline{\Omega}$, *where* A *is defined in (6).*

Proof. Under the above hypotheses, Choi and McKenna proved in [3] that every solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (9), with $u > 0$ in Ω, satisfies

$$\varphi \le u \le \psi \quad \text{in } \overline{\Omega}.$$

Moreover, if only conditions $(\mathbf{A}_1) - (\mathbf{A}_3)$ and $(\mathbf{U})$ hold, then $u \le \psi$ in $\overline{\Omega}$.

It is easy to check that the function ψ defined in (4) satisfies condition $(\mathbf{U})$ considered for our problem (3). Therefore, by the Choi-McKenna comparison lemma and (5), we find that every positive classical solution of (3) is bounded above by the same number A defined in (6). □

3 Proof of Theorem 1

Let u and v be solutions of (3) and let u satisfy (7), where

$$K_1 = \frac{\pi^2}{\ell^2} \frac{1}{f_1(A)} \inf_{(0,A)} \frac{f_1^2}{f_1'} + \alpha \inf_{(0,A)} \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}.$$

We prove in what follows that $u = v$ in $\overline{\Omega}$. Set

$$w(x) = \frac{u(x', y)}{s(y)}, \quad z(x) = \frac{v(x', y)}{s(y)},$$

where

$$s(y) = \sin \frac{\pi y}{\ell}, \quad c(y) = \cos \frac{\pi y}{\ell} \qquad y \in (0, \ell).$$

Since $s > 0$ and $s \in C^\infty$, it follows that w and z are well defined and they are as smooth as u and v respectively on Ω. A simple computation shows that w satisfies the boundary value problem

$$\begin{cases} \displaystyle\sum_{i=1}^{N-1} s f_i(u) w_{x_i x_i} + \frac{2\pi c}{\ell} w_y + s w_{yy} - \frac{\pi^2 s}{\ell^2} w + p(x) g(u) = 0, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Similarly,

$$\begin{cases} \displaystyle\sum_{i=1}^{N-1} s f_i(v) z_{x_i x_i} + \frac{2\pi c}{\ell} z_y + s z_{yy} - \frac{\pi^2 s}{\ell^2} z + p(x) g(v) = 0, & \text{in } \Omega \\ z = 0, & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Relations (10) and (11) yield

$$\sum_{i=1}^{N-1} s \frac{f_i(v)}{f_1(v)} (z-w)_{x_i x_i} + \sum_{i=2}^{N-1} s \left[\left(\frac{f_i}{f_1}\right)(v) - \left(\frac{f_i}{f_1}\right)(u) \right] w_{x_i x_i} + \frac{2\pi c}{\ell} \frac{1}{f_1(v)} (z-w)_y +$$
$$\frac{2\pi c}{\ell} \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) w_y + s \frac{1}{f_1(v)} (z-w)_{yy} + s \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) w_{yy} -$$
$$\frac{\pi^2 s}{\ell^2} \frac{1}{f_1(v)} (z-w) - \frac{\pi^2 s}{\ell^2} \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) w + p(x) \left[\left(\frac{g}{f_1}\right)(v) - \left(\frac{g}{f_1}\right)(u) \right] = 0.$$

Whenever $z \neq w$ we can rewrite the above equation as follows

$$\sum_{i=1}^{N-1} s\frac{f_i(v)}{f_1(v)}(z-w)_{x_ix_i} + s\frac{1}{f_1(v)}(z-w)_{yy} + \frac{2\pi c}{\ell}\frac{1}{f_1(v)}(z-w)_y + \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right)Q(z,w) = 0 \tag{12}$$

where

$$Q(z,w) = u_{yy} + \sum_{i=2}^{N-1} \frac{\left(\frac{f_i}{f_1}\right)(v) - \left(\frac{f_i}{f_1}\right)(u)}{\left(\frac{1}{f_1}\right)(v) - \left(\frac{1}{f_1}\right)(u)} u_{x_ix_i} - \frac{\pi^2}{\ell^2}\frac{1}{f_1(v)}\frac{v-u}{\left(\frac{1}{f_1}\right)(v) - \left(\frac{1}{f_1}\right)(u)} + p(x)\frac{\left(\frac{g}{f_1}\right)(v) - \left(\frac{g}{f_1}\right)(u)}{\left(\frac{1}{f_1}\right)(v) - \left(\frac{1}{f_1}\right)(u)}.$$

In order to conclude the proof it is enough to show that

$$Q(z,w) > 0 \quad \text{whenever } z \neq w. \tag{13}$$

Indeed, if $(z-w) > 0$ at some point in Ω, then $\max\limits_{\overline{\Omega}}(z-w)$ is achieved in Ω, since $z = w = 0$ on $\partial\Omega$. At that point we have

$$(z-w)_{x_ix_i} \leq 0, \quad (z-w)_{yy} \leq 0, \quad (z-w)_y = 0 \quad \text{and} \quad \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right)Q(z,w) < 0$$

which contradicts (12). A similar argument shows that $(z-w)$ cannot be negative at any point in Ω. Hence $z = w$ in Ω which implies $u = v$ on $\overline{\Omega}$.

For every $x \in \Omega$, let us define

$$\mu(x) = \min(u(x), v(x)) \quad \text{and} \quad \nu(x) = \max(u(x), v(x))\,.$$

Thus, by Lemma 1, $\nu \leq A$ in Ω.

In (12) we apply the Cauchy generalized mean value theorem on every interval $[\mu(x), \nu(x)]$ where $x \in \Omega$ is taken such that $z(x) \neq w(x)$. Hence, for all $i = \overline{2, N-1}$ we obtain the existence of $\xi_i(x)$, $\sigma(x)$, $\lambda(x) \in (\mu(x), \nu(x)) \subset (0, A)$ such that

$$m_i \leq \frac{\left(\frac{f_i}{f_1}\right)(v(x)) - \left(\frac{f_i}{f_1}\right)(u(x))}{\left(\frac{1}{f_1}\right)(v(x)) - \left(\frac{1}{f_1}\right)(u(x))} = \frac{\left(\frac{f_i}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}(\xi_i(x)) \leq M_i \tag{14}$$

$$-\frac{v(x) - u(x)}{\left(\frac{1}{f_1}\right)(v(x)) - \left(\frac{1}{f_1}\right)(u(x))} = \frac{f_1^2}{f_1'}(\sigma(x)) \geq \inf_{(0,A)} \frac{f_1^2}{f_1'} \tag{15}$$

$$\frac{\left(\frac{g}{f_1}\right)(v(x)) - \left(\frac{g}{f_1}\right)(u(x))}{\left(\frac{1}{f_1}\right)(v(x)) - \left(\frac{1}{f_1}\right)(u(x))} = \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}(\lambda(x)) \geq \inf_{(0,A)} \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}. \tag{16}$$

Using (14), (15) and (16) we find

$$Q(z,w) \geq u_{yy} + \sum_{i \in P_x} m_i u_{x_ix_i} + \sum_{i \in N_x} M_i u_{x_ix_i} + \frac{\pi^2}{\ell^2}\frac{1}{f_1(A)} \inf_{(0,A)} \frac{f_1^2}{f_1'} + \alpha \inf_{(0,A)} \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'} =$$

$$u_{yy} + \sum_{i \in P_x} m_i u_{x_ix_i} + \sum_{i \in N_x} M_i u_{x_ix_i} + K_1.$$

Since the solution u satisfies (7) we obtain that relation (13) is true. This completes the proof. $\square$

4 Proof of Theorem 2

Let u and v be two solutions of (3) and set

$$K_2 = -\alpha \sup_{(0,A)} \frac{g'}{f_1'} \geq 0\,. \tag{17}$$

The functions w, z, μ and ν will have the same signification as in the above proof.

By (10) and (11) it follows that

$$\begin{aligned}&\sum_{i=1}^{N-1} sf_i(v)(z-w)_{x_ix_i} + \sum_{i=1}^{N-1} s[f_i(v)-f_i(u)]w_{x_ix_i} + \frac{2\pi c}{\ell}(z-w)_y + s(z-w)_{yy} - \\ &\frac{\pi^2 s}{\ell^2}(z-w) + p(x)[g(v)-g(u)] = 0.\end{aligned} \tag{18}$$

Whenever $z \neq w$, relation (18) may be rewritten in the following form

$$\sum_{i=1}^{N-1} sf_i(v)(z-w)_{x_ix_i} + \frac{2\pi c}{\ell}(z-w)_y + s(z-w)_{yy} + [f_1(v)-f_1(u)]R(z,w) = 0\,,$$

where

$$R(z,w) = u_{x_1x_1} + \sum_{i=2}^{N-1} \frac{f_i(v)-f_i(u)}{f_1(v)-f_1(u)} u_{x_ix_i} - \frac{\pi^2}{\ell^2}\frac{v-u}{f_1(v)-f_1(u)} + p(x)\frac{g(v)-g(u)}{f_1(v)-f_1(u)}.$$

Using the maximum principle (as we did in the proof of Theorem 1) we see that the proof will be concluded if we prove that

$$R(z,w) < 0 \quad \text{whenever } z \neq w.$$

From now on, we shall consider only the points $x \in \Omega$ with the property that $z(x) \neq w(x)$. For these points, we apply again the Cauchy generalized mean value theorem on $[\mu(x), \nu(x)]$ and we obtain $\eta_i(x)$, $\theta(x)$, $\zeta(x) \in (\mu(x), \nu(x)) \subset (0, A)$ such that

$$\frac{f_i(v(x))-f_i(u(x))}{f_1(v(x))-f_1(u(x))} = \frac{f_i'}{f_1'}(\eta_i(x)) \geq \inf_{(0,A)} \frac{f_i'}{f_1'}, \quad i = \overline{2, N-1} \tag{19}$$

$$\frac{v(x)-u(x)}{f_1(v(x))-f_1(u(x))} = \frac{1}{f_1'(\theta(x))} \tag{20}$$

$$\frac{g(v(x))-g(u(x))}{f_1(v(x))-f_1(u(x))} = \frac{g'}{f_1'}(\zeta(x)) \leq \sup_{(0,A)} \frac{g'}{f_1'} \leq 0. \tag{21}$$

It is easy to verify that hypothesis ($\mathbf{C}_2$) implies

$$\frac{f_i(v(x))-f_i(u(x))}{f_1(v(x))-f_1(u(x))} \leq \frac{f_i(u(x))}{f_1(u(x))} \quad \text{for all } i = \overline{2, N-1}. \tag{22}$$

On the other hand, since f_1 is increasing on $(0, A)$,

$$\frac{v(x)-u(x)}{f_1(v(x))-f_1(u(x))} > 0. \tag{23}$$

Combining relations (19), (21), (22) and (23) with the expression of $R(z,w)$ we deduce that

$$R(z,w) < u_{x_1x_1} + \sum_{i\in P_x} \frac{f_i(u)}{f_1(u)} u_{x_ix_i} + \sum_{i\in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_ix_i} + \alpha \sup_{(0,A)} \frac{g'}{f_1'} .$$

Since u is a solution of (3) satisfying (8) we deduce that $R(z,w)$ is negative. This completes our proof. □

As an application of this result, let us consider the problem

$$\begin{cases} f(u)u_{x_1x_1} + \displaystyle\sum_{i=2}^{N} u_{x_ix_i} + p(x) = 0, & \text{if } x \in B(0,1) \subset \mathbf{R}^N \\ u = 0, & \text{if } |x| = 1, \end{cases} \tag{24}$$

where $f : (0,\infty) \to (0,\infty)$ is a given increasing function and $p(x) = 2(N-1) + 2f(1-|x|^2)$. We first observe that the function $u(x) = 1 - |x|^2$ is a solution of (24). In order to establish its uniqueness, observing first that $u_{x_ix_i} = -2$, for any $i = 1, \cdots, N$, we deduce that condition (7) reduces to $K_1 > 2N$, which depends on f. For instance, if $f(u) = u^p$, $p > 0$, then $K_1 = 2N - 2$, hence Theorem 1 does not apply. However we observe that conditions ($\mathbf{H}_1$)-($\mathbf{H}_3$) and ($\mathbf{C}_2$) are fulfilled. Furthermore, (17) yields $K_2 = 0$. It follows that assumption (8) is automatically satisfied in our case since the left hand-side of (8) equals to -2. Therefore, by Theorem 2, problem (24) has a unique solution.

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