GENERALISATIONS OF A CERTAIN INEQUALITY USED IN THE THEORY OF DIFFERENTIAL EQUATIONS

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Abstract. In the present paper we establish some new inequalities related to a certain integral inequality used in the theory of differential equations. An application to obtain the bound on the solutions of a certain integrodifferential equation is also given.

1. Introduction

The following inequality is used considerably in the theory of differential equations.

Theorem 1. *Let y and f be real-valued nonnegative continuous functions defined for $t \in \mathbb{R}_+ = [0, \infty)$. If*

$$y^2(t) \leq c^2 + 2\int_0^t f(s)\, y(s)\, ds,$$

for $t \in \mathbb{R}_+$, where $c \geq 0$ is a constant, then

$$y(t) \leq c + \int_0^t f(s)\, ds,$$

for $t \in \mathbb{R}_+$.

The following variant of the above inequality is used by Dafermos [4] to establish a different connection between stability and the second law of thermodynamics.

Theorem 2. *Assume that the nonnegative function $y(t) \in L^\infty[0, s]$ and $g(t) \in L^1[0, s]$ satisfy the inequality*

$$y^2(t) \leq M^2 y^2(0) + \int_0^t \left[2\alpha y^2(s) + 2N g(s)\, y(s)\right] ds, \quad t \in [0, s],$$

where α, M, N are nonnegative constants. Then

$$y(s) \leq M e^{\alpha s} y(0) + N e^{\alpha s} \int_0^s g(t)\, dt.$$

In the past few years the inequality given in Theorem 1 has been used to obtain global existence, uniqueness, stability and other properties of the solutions for wide classes of nonlinear differential equations (see [1], [2], [5]-[9], [15]). Recently, in [11]- [14] the present author established a number of new integral and discrete inequalities related to the inequalities given in Theorems 1 and 2. The aim of the present paper is to establish some new generalisations of the inequalities given in

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Theorems 1 and 2 which can be used as tools in the analysis of certain new classes of differential and integral equations. The discrete analogues of the main results are also given. Finally, an application to convey the importance of our results to the literature is given.

2. Statement of Results

In this section we state our main results to be proved in this paper. In what follows, we let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}$ the set of real numbers and $N_0 = \{0, 1, 2, \dots\}$. For $m > n$, $m, n \in N_0$ and any function $q(n)$ defined on N_0 we use the usual conventions $\sum_{s=m}^{n} q(s) = 0$ and $\prod_{s=m}^{n} q(s) = 1$.

Our main result is embodied in the following theorem.

Theorem 3. *Let y, f, g, h, k be real-valued nonnegative continuous functions defined on $\mathbb{R}_+$ and c be a nonnegative real constant. If*

$$\begin{aligned} &y^2(t) \\ &\leq c^2 + 2\int_0^t y(s)\left[f(s)\,y(s) + g(s) + h(s)\left(y(s) + \int_0^s k(\tau)\,y(\tau)\,d\tau\right)\right] ds, \end{aligned} \tag{2.1}$$

for $t \in \mathbb{R}_+$, then

$$y(t) \leq c\exp\left(\int_0^t f(s)\,ds\right) + \int_0^t [g(s) + h(s)\,A(s)]\exp\left(\int_s^t f(\tau)\,d\tau\right) ds, \tag{2.2}$$

for $t \in \mathbb{R}_+$, where

$$\begin{aligned} A(t) &= c\exp\left(\int_0^t [f(\tau) + h(\tau) + k(\tau)]\,d\tau\right) \\ &\quad + \int_0^t g(\tau)\exp\left(\int_\tau^t [f(\eta) + h(\eta) + k(\eta)]\,d\eta\right) d\tau, \end{aligned} \tag{2.3}$$

for $t \in \mathbb{R}_+$.

A slight variant of Theorem 3 is given in the following theorem.

Theorem 4. *Let y, f, g, h, k, c be as in Theorem 3. Then the inequality (2.1) implies*

$$\begin{aligned} y(t) \leq\ & p(t)\left[\exp\left(\int_0^t f(s)\,ds\right) + \int_0^t h(s)\right. \\ & \left.\times\exp\left(\int_0^s [f(\tau) + h(\tau) + k(\tau)]\,d\tau\right)\exp\left(\int_s^t f(\tau)\,d\tau\right) ds\right], \end{aligned} \tag{2.4}$$

for $t \in \mathbb{R}_+$, where

$$p(t) = c + \int_0^t g(s)\,ds, \tag{2.5}$$

for $t \in \mathbb{R}_+$.

The discrete analogues of Theorems 3 and 4 are given in the following theorems.

Theorem 5. *Let y, f, g, h, k be real-valued nonnegative functions defined on N_0 and c be a nonnegative real constant. If*

$$y^2(n) \leq c^2 + 2\sum_{s=0}^{n-1} y(s)\left[f(s)\,y(s) + g(s) + h(s)\left(y(s) + \sum_{t=0}^{s-1} k(t)\,y(t)\right)\right] \tag{2.6}$$

for $n \in N_0$, *then*

$$y(n) \leq c \prod_{s=0}^{n-1} [1 + f(s)] + \sum_{s=0}^{n-1} [g(s) + h(s) B(s)] \sum_{t=s+1}^{n-1} [1 + f(t)], \tag{2.7}$$

for $n \in N_0$, *where*

$$\begin{aligned} B(n) &= c \prod_{\tau=0}^{n-1} [1 + f(\tau) + h(\tau) + k(\tau)] \\ &\quad + \sum_{\tau=0}^{n-1} g(\tau) \prod_{t=\tau+1}^{n-1} [1 + f(t) + h(t) + k(t)], \end{aligned} \tag{2.8}$$

for $n \in N_0$.

Theorem 6. *Let* y, f, g, h, k, c *be as in Theorem 5. Then the inequality (2.6) implies that*

$$\begin{aligned} y(n) &\leq q(n) \Bigg[\prod_{s=0}^{n-1} [1 + f(s)] \\ &\quad + \sum_{s=0}^{n-1} h(s) \prod_{t=0}^{s-1} [1 + f(t) + h(t) + k(t)] \prod_{\tau=s+1}^{n-1} [1 + f(\tau)] \Bigg], \end{aligned} \tag{2.9}$$

for $n \in N_0$, *where*

$$q(n) = c + \sum_{s=0}^{n-1} g(s), \tag{2.10}$$

for $n \in N_0$.

3. Proofs of Theorems 3 and 4

Proof of Theorem 3. In order to establish the inequality (2.1) in Theorem 3, we first assume that $c > 0$ and define a function $z(t)$ by

$$\begin{aligned} & z(t) \\ = \; & c^2 + 2 \int_0^t y(s) \left[f(s) y(s) + g(s) + h(s) \left(y(s) + \int_0^s k(\tau) y(\tau) d\tau \right) \right] ds. \end{aligned} \tag{3.1}$$

Differentiating (3.1) and then using the fact that $y(t) \leq \sqrt{z(t)}$ we have

$$\begin{aligned} & z'(t) \\ \leq \; & 2\sqrt{z(t)} \left[f(t) \sqrt{z(t)} + g(t) + h(t) \left(\sqrt{z(t)} + \int_0^s k(\tau) \sqrt{z(\tau)} d\tau \right) \right]. \end{aligned} \tag{3.2}$$

Now differentiating $\sqrt{z(t)}$ and using (3.2) we have

$$\begin{aligned} & \frac{d}{dt} \left(\sqrt{z(t)} \right) \\ = \; & \frac{z'(t)}{2\sqrt{z(t)}} \\ \leq \; & \left[f(t) \sqrt{z(t)} + g(t) + h(t) \left(\sqrt{z(t)} + \int_0^s k(\tau) \sqrt{z(\tau)} d\tau \right) \right]. \end{aligned} \tag{3.3}$$

Define a function $v(t)$ by

$$v(t) = \sqrt{z(t)} + \int_0^t k(\tau)\sqrt{z(\tau)}d\tau. \tag{3.4}$$

Differentiating (3.4) and then using (3.3) and the fact that $\sqrt{z(t)} \leq v(t)$, we have

$$v'(t) \leq [f(t) + h(t) + k(t)]\, v(t) + g(t). \tag{3.5}$$

The inequality (3.5) implies the estimate

$$v(t) \leq A(t). \tag{3.6}$$

Using (3.6) in (3.3) we have

$$\frac{d}{dt}\left(\sqrt{z(t)}\right) \leq f(t)\sqrt{z(t)} + g(t) + h(t)\, A(t). \tag{3.7}$$

The inequality (3.7) implies the estimate

$$\begin{aligned} \sqrt{z(t)} \quad &\leq \quad c\exp\left(\int_0^t f(s)\, ds\right) \\ &\quad + \int_0^t [g(s) + h(s)\, A(s)] \exp\left(\int_s^t f(\tau)\, d\tau\right) ds. \end{aligned} \tag{3.8}$$

Now, using the fact that $y(t) \leq \sqrt{z(t)}$ in (3.8) we get the required inequality in (2.1).

If c is nonnegative, we can carry out the above procedure with $c + \varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.1).This completes the proof of Theorem 3. ∎

Proof of Theorem 4. By following the proof of Theorem 3 given above we have (3.3). By setting $t = s$ in (3.3) and then integrating with respect to s from 0 to t we have

$$\begin{aligned} &\sqrt{z(t)} \\ &\leq \quad p(t) + \int_0^t \left[f(s)\sqrt{z(s)} + h(s)\left(\sqrt{z(s)} + \int_0^s k(\tau)\sqrt{z(\tau)}d\tau\right)\right] ds. \end{aligned} \tag{3.9}$$

Since $p(t)$ is positive and monotone non-decreasing in $t \in \mathbb{R}_+$, from (3.9) we observe that

$$\begin{aligned} &\frac{\sqrt{z(t)}}{p(t)} \\ &\leq \quad 1 + \int_0^t \left[f(s)\frac{\sqrt{z(s)}}{p(s)} + h(s)\left(\frac{\sqrt{z(s)}}{p(s)} + \int_0^s k(\tau)\frac{\sqrt{z(\tau)}}{p(\tau)}d\tau\right)\right] ds. \end{aligned} \tag{3.10}$$

Define a function $r(t)$ by the right side of (3.10). Then differentiating $r(t)$ and using the fact that $\frac{\sqrt{z(t)}}{p(t)} \leq r(t)$ we have

$$r'(t) \leq f(t)\, r(t) + h(t)\left(r(t) + \int_0^t k(\tau)\, r(\tau)\, d\tau\right). \tag{3.11}$$

Define a function $w(t)$ by

$$w(t) = r(t) + \int_0^t k(\tau)\, r(\tau)\, d\tau. \tag{3.12}$$

Differentiating (3.2) and using (3.11) and the fact that $r(t) \leq w(t)$ we have

$$w'(t) \leq [f(t) + h(t) + k(t)]\, r(t)\,. \tag{3.13}$$

The inequality (3.13) implies the estimate

$$w(t) \leq \exp\left(\int_0^t [f(\tau) + h(\tau) + k(\tau)]\, d\tau\right). \tag{3.14}$$

Using (3.14) in (3.11) we have

$$r'(t) \leq f(t)\, r(t) + h(t) \exp\left(\int_0^t [f(\tau) + h(\tau) + k(\tau)]\, d\tau\right). \tag{3.15}$$

The inequality (3.15) implies the estimate

$$\begin{aligned} r(t) \quad \leq \quad & \exp\left(\int_0^t f(s)\, ds\right) + \int_0^t h(s) \\ & \times \exp\left(\int_0^s [f(\tau) + h(\tau) + k(\tau)]\, d\tau\right) \exp\left(\int_s^t f(\tau)\, d\tau\right) ds. \end{aligned} \tag{3.16}$$

Using (3.16) in (3.10) and the fact that $y(t) \leq \sqrt{z(t)}$, we get the desired inequality in (2.4). The proof of the case when c is nonnegative cam be completed as mentioned in the proof of Theorem 3. This completes the proof of Theorem 4. ∎

Remark 1. *We note that, from the proof of Theorem 3 given above we have (3.6). By using the facts that $\sqrt{z(t)} \leq v(t)$ and $y(t) \leq \sqrt{z(t)}$ in (3.6) we get the following inequality*

$$y(t) \leq A(t)\,, \tag{3.17}$$

for $t \in \mathbb{R}_+$, where $A(t)$ is defined in (2.3). Thus, in this case (3.17) gives a simpler but not necessarily smaller bound than (2.2). If we take

(i) $f = h = k = 0$ *and*
(ii) $h = k = 0$,

then the inequalities obtained in Theorems 3, 4 and (3.17) reduce respectively to the inequalities given in Theorem 1 and a further generalisation of the inequality given in Theorem 2.

4. Proofs of Theorems 5 and 6

Proof. Assume that c is positive and define a function $z(n)$ by

$$z(n) \leq c^2 + 2\sum_{s=0}^{n-1} y(s)\left[f(s)\, y(s) + g(s) + h(s)\left(y(s) + \sum_{t=0}^{s-1} k(t)\, y(t)\right)\right]. \tag{4.1}$$

From (4.1) and using the fact that $y(n) \leq \sqrt{z(n)}$ we observe that

$$\begin{aligned} z(n+1) - z(n) \quad \leq \quad & 2\sqrt{z(n)}\Big[f(n)\sqrt{z(n)} + g(n) \\ & + h(n)\left(\sqrt{z(n)} + \sum_{t=0}^{n-1} k(t)\sqrt{z(t)}\right)\Big]. \end{aligned} \tag{4.2}$$

It is easy to observe that

$$\sqrt{z(n+1)} - \sqrt{z(n)} = \frac{z(n+1) - z(n)}{\sqrt{z(n+1)} + \sqrt{z(n)}} \leq \frac{z(n+1) - z(n)}{2\sqrt{z(n)}} \tag{4.3}$$

Here in the last step of (4.3), we have used the fact that $\sqrt{z(n)} \leq \sqrt{z(n+1)}$. By using (4.2) in (4.3) we get

$$\sqrt{z(n+1)} - \sqrt{z(n)} \leq \left[f(n)\sqrt{z(n)} + g(n) + h(n)\left(\sqrt{z(n)} + \sum_{t=0}^{n-1} k(t)\sqrt{z(t)} \right) \right]. \tag{4.4}$$

Define a function $v(n)$ by

$$v(n) = \sqrt{z(n)} + \sum_{t=0}^{n-1} k(t)\sqrt{z(t)}. \tag{4.5}$$

From (4.5) and using (4.4) and the fact that $\sqrt{z(n)} \leq v(n)$ we observe that

$$v(n+1) - [1 + f(n) + h(n) + k(n)]\, v(n) \leq g(n). \tag{4.6}$$

The inequality (4.6) implies the estimate (see [10])

$$v(n) \leq B(n). \tag{4.7}$$

Using (4.7) in (4.4), we observe that

$$\sqrt{z(n+1)} - [1 + f(n)]\sqrt{z(n)} \leq g(n) + h(n)\, B(n). \tag{4.8}$$

The inequality (4.8) implies the estimate (see [10])

$$\sqrt{z(n)} \leq c \prod_{s=0}^{n-1} [1 + f(s)] + \prod_{s=0}^{n-1} [g(s) + h(s)\, B(s)] \sum_{t=s+1}^{n-1} [1 + f(t)]. \tag{4.9}$$

Now, by using the fact that $y(n) \leq \sqrt{z(n)}$ in (4.9), we get the required inequality in (2.7). The proof of the case when c is nonnegative can be completed as mentioned in the proof of Theorem 3. This completes the proof of Theorem 5. ∎

The proof of Theorem 6 can be completed by following the proof of Theorem 5 and closely looking at the proof of Theorem 4 given above with suitable modifications. Here we omit the details.

Remark 2. *We note that, from the proof of Theorem 5 we have (4.7). Using the facts that* $\sqrt{z(n)} \leq v(n)$ *and* $y(n) \leq \sqrt{z(n)}$ *in (4.7), we get the following inequality*

$$y(n) \leq B(n), \tag{4.10}$$

for $n \in N_0$*, where* $B(n)$ *is defined by (2.8). If we take*

(i) $f = h = k = 0$ *and*
(ii) $h = k = 0$,

then the inequalities obtained in Theorems 5 and 6 and (4.10) reduce respectively the discrete analogue of the inequality given in Theorem 1 and the discrete generalisation of the inequality given in Theorem 2.

5. An Application

In this section, we present an application of our Theorem 4 to obtain bounds on the solutions of the following integrodifferential equation

$$x'(t) = F(t, x(t), \sigma(t)), \quad x(0) = x_0, \tag{5.1}$$

with

$$\sigma(t) = \phi(t) + G\left(t, x(t), \int_0^t H(t, \tau, x(\tau))\, d\tau\right), \tag{5.2}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$, $H : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. For an excellent account on the study of equations of the above type, see the monograph by C. Corduneanu [3]. Here we assume that the solution $x(t)$ of (5.1) exists on $\mathbb{R}_+$. Multiplying both sides of equation (5.1) by $x(t)$, substituting $t = s$ and then integrating it from 0 to t, we have

$$x^2(t) = x_0^2 + 2\int_0^t x(s)\, F(s, x(s), \sigma(s))\, ds. \tag{5.3}$$

We assume that

$$\begin{aligned} |H(t, \tau, x(\tau))| &\le k(\tau)\, |x(\tau)|, & (5.4)\\ |G(t, x(t), u)| &\le h(t)\, [|x(t)| + |u|], & (5.5)\\ |F(t, x(t), \sigma(t))| &\le f(t)\, |x(t)| + |\sigma(t)|, & (5.6) \end{aligned}$$

where k, h, f are real-valued nonnegative continuous functions defined on $\mathbb{R}_+$. From (5.3) - (5.6) we observe that

$$\begin{aligned} |x(t)|^2 \le{}& |x_0|^2 + 2\int_0^t |x(s)| \Bigg[f(s)\, |x(s)| + |\phi(s)| \\ &+ h(s) \left(|x(s)| + \int_0^s k(\tau)\, |x(\tau)|\, d\tau \right) \Bigg] ds. \end{aligned} \tag{5.7}$$

Now an application of Theorem 4 yields

$$\begin{aligned} &|x(t)| \\ \le{}& p_1(t) \Bigg[\exp\left(\int_0^t f(s)\, ds \right) + \int_0^t h(s) \\ &\times \exp\left(\int_0^s [f(\tau) + h(\tau) + k(\tau)]\, d\tau \right) \exp\left(\int_s^t f(\tau)\, d\tau \right) ds \Bigg], \end{aligned} \tag{5.8}$$

for all $t \in \mathbb{R}_+$, where

$$p_1(t) = |x_0| + \int_0^t |\phi(s)|\, ds$$

The inequality (5.8) gives the bound on the solution $x(t)$ of (5.1) in terms of the known functions.

Remark 3. *We note that the inequalities established in Theorems 5 and 6 can be used to obtain bounds on the solutions of the following sum-difference equation*

$$\Delta z^2(n) = 2z(n)\, F(n, z(n), \sigma(n)), \quad z(0) = z_0, \tag{5.9}$$

with

$$\sigma(n) = \phi(n) + G\left(n, z(n), \sum_{S=0}^{n-1} H(n, s, z(s))\right), \tag{5.10}$$

under some suitable conditions on the functions involved in (5.9), (5.10), where Δ *is the forward difference operator. For similar applications, see* [12] *and* [14]*. We also note that the inequalities obtained in Theorems 3 - 6 can very easily be extended to functions of several independent variables. The multidimensional versions of these inequalities along with various other applications will be reported elsewhere.*

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