GRÜSS INEQUALITY IN TERMS OF $\Delta-$SEMINORMS AND APPLICATIONS

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Abstract. Some upper bounds for the modulus of the Chebychev functional in terms of $\Delta-$seminorms are pointed out. Applications for midpoint and trapezoid inequalities are also given.

1. Introduction

For two measurable functions $f,g:[a,b]\to\mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f,g;a,b):=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx, \tag{1.1}$$

provided that the involved integrals exist.

The following inequality is well known in the literature as the Grüss inequality [9]

$$|T(f,g;a,b)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that $m\le f\le M$ and $n\le g\le N$ a.e. on $[a,b]$, where m,M,n,N are real numbers. The constant $\frac{1}{4}$ in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [1, p. 207]). Namely, if f,g are absolutely continuous on $[a,b]$ and $f',g'\in L_\infty[a,b]$ and $\|f'\|_\infty:=ess\sup_{t\in[a,b]}|f'(t)|$, then

$$|T(f,g;a,b)|\le\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2 \tag{1.3}$$

and the constant $\frac{1}{12}$ is the best possible.

Finally, let us recall a result by Lupaş (see for example [1, p. 210]), which states that:

$$|T(f,g;a,b)|\le\frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a)^2, \tag{1.4}$$

provided f,g are absolutely continuous and $f',g'\in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible here.

For other Grüss type inequalities, see the books [1] and [2], and the papers [3]-[10], where further references are given.

In the present paper we point out some bounds for the Chebychev functional in terms of the $\Delta-$seminorms $\|\cdot\|_p^\Delta$, $p\in[1,\infty]$; as will be defined in the sequel.

Date: June 16, 2000.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20, 26D99; Secondary 41A55.

Key words and phrases. Grüss, Chebychev and Lupaş inequalities, Numerical Analysis.

2. $\Delta-$Seminorms and Related Inequalities

For $f \in L_p[a,b]$ ($p \in [1,\infty)$) we can define the functional (see also [11])

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b |f(t) - f(s)|^p \, dtds \right)^{\frac{1}{p}} \tag{2.1}$$

and for $f \in L_\infty[a,b]$, we can define

$$\|f\|_\infty^{\Delta} := ess \sup_{(t,s)\in[a,b]^2} |f(t) - f(s)| . \tag{2.2}$$

If we consider $f_\Delta : [a,b]^2 \to \mathbb{R}$,

$$f_\Delta(t,s) = f(t) - f(s), \tag{2.3}$$

then, obviously

$$\|f\|_p^{\Delta} = \|f_\Delta\|_p, \ \ p \in [1,\infty], \tag{2.4}$$

where $\|\cdot\|_p$ are the usual Lebesgue p-norms on $[a,b]^2$.

Using the properties of the Lebesgue $p-$norms, we may deduce the following semi-norm properties for $\|\cdot\|_p^{\Delta}$:

(*i*) $\|f\|_p^{\Delta} \geq 0$ for $f \in L_p[a,b]$ and $\|f\|_p^{\Delta} = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$;

(*ii*) $\|f+g\|_p^{\Delta} \leq \|f\|_p^{\Delta} + \|g\|_p^{\Delta}$ if $f,g \in L_p[a,b]$;

(*iii*) $\|\alpha f\|_p^{\Delta} = |\alpha| \, \|f\|_p^{\Delta}$.

We note that if $p = 2$, then,

$$\begin{aligned} \|f\|_2^{\Delta} &= \left(\int_a^b \int_a^b (f(t) - f(s))^2 \, dtds \right)^{\frac{1}{2}} \\ &= \sqrt{2} \left[(b-a) \|f\|_2^2 - \left(\int_a^b f(t) \, dt \right)^2 \right]^{\frac{1}{2}} . \end{aligned}$$

Using the inequalities (1.2), (1.3) and (1.4), we obtain the following estimate for $\|\cdot\|_2^{\Delta}$:

$$\|f\|_2^{\Delta} \leq \begin{cases} \frac{\sqrt{2}}{2}(M-m) & \text{if} \quad m \leq f \leq M; \\ \frac{\sqrt{2}}{2\sqrt{3}} \|f'\|_\infty (b-a) & \text{if} \quad f' \in L_\infty[a,b]; \\ \frac{\sqrt{2}}{\pi} \|f'\|_2 (b-a) & \text{if} \quad f' \in L_2[a,b]. \end{cases}$$

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^{\Delta}$ in terms of $\|f'\|_p$.

Theorem 1. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

(*i*) *If* $p \in [1, \infty)$, *then we have the inequality*

$$\|f\|_p^{\Delta} \leq \begin{cases} \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty & \text{if} \quad f' \in L_\infty[a,b]; \\ \frac{(2\beta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha[a,b], \\ & \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)^{\frac{2}{p}} \|f'\|_1 & \text{if} \quad f' \in L_1[a,b], \end{cases} \tag{2.5}$$

(*ii*) *If* $p = \infty$, *then we have the inequality*

$$\|f\|_\infty^{\Delta} \leq \begin{cases} (b-a)\|f'\|_\infty & \text{if} \quad f' \in L_\infty[a,b]; \\ (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha[a,b], \ \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1 . \end{cases} \tag{2.6}$$

Proof. As $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then $f(t) - f(s) = \int_s^t f'(u)\,du$ for all $t, s \in [a,b]$, and then

$$\begin{aligned} & |f(t) - f(s)| \\ = & \left| \int_s^t f'(u)\,du \right| \leq \begin{cases} |t-s| \|f'\|_\infty & \text{if} \quad f' \in L_\infty[a,b]; \\ |t-s|^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha[a,b], \ \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1 & \text{if} \quad f' \in L_1[a,b] \end{cases} \end{aligned} \tag{2.7}$$

and so for $p \in [1, \infty)$, we may write

$$\begin{aligned} & |f(t) - f(s)|^p \\ \leq & \begin{cases} |t-s|^p \|f'\|_\infty^p & \text{if} \quad f' \in L_\infty[a,b]; \\ |t-s|^{\frac{p}{\beta}} \|f'\|_\alpha^p & \text{if} \quad f' \in L_\alpha[a,b], \ \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1^p & \text{if} \quad f' \in L_1[a,b], \end{cases} \end{aligned}$$

and then from (2.3), (2.4)

$$\|f\|_p^{\Delta} \leq \begin{cases} \|f'\|_\infty \left(\int_a^b \int_a^b |t-s|^p \,dtds \right)^{\frac{1}{p}} & \text{if} \quad f' \in L_\infty[a,b]; \\ \|f'\|_\alpha \left(\int_a^b \int_a^b |t-s|^{\frac{p}{\beta}} \,dtds \right)^{\frac{1}{p}} & \text{if} \quad f' \in L_\alpha[a,b], \\ & \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1 \left(\int_a^b \int_a^b dtds \right)^{\frac{1}{p}} & \text{if} \quad f' \in L_1[a,b]. \end{cases} \tag{2.8}$$

Further, since

$$\begin{aligned}(2.9)\qquad & \left(\int_a^b \int_a^b |t-s|^p \, dtds\right)^{\frac{1}{p}} \\ = & \left[\int_a^b \left(\int_a^t (t-s)^p \, ds + \int_t^b (s-t)^p \, ds\right) dt\right]^{\frac{1}{p}} \\ = & \left(\int_a^b \left[\frac{(t-a)^{p+1} + (b-t)^{p+1}}{p+1}\right] dt\right)^{\frac{1}{p}} \\ = & \frac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}},\end{aligned}$$

giving

$$\left(\int_a^b \int_a^b |t-s|^{\frac{p}{\beta}} \, dtds\right)^{\frac{1}{p}} = \frac{(2\beta^2)^{\frac{1}{p}} (b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}},$$

and

$$\left(\int_a^b \int_a^b dtds\right)^{\frac{1}{p}} = (b-a)^{\frac{2}{p}},$$

we obtain, from (2.8), the stated result (2.5).

Using (2.7) we have (for $p = \infty$) that

$$(2.10)\qquad \|f\|_\infty^\Delta \le \begin{cases} \|f'\|_\infty \, ess \sup_{(t,s)\in[a,b]^2} |t-s| \\ \\ \|f'\|_\alpha \, ess \sup_{(t,s)\in[a,b]} |t-s|^{\frac{1}{\beta}} \\ \\ \|f'\|_1 \end{cases} = \begin{cases} (b-a)\|f'\|_\infty \\ \\ (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha \\ \\ \|f'\|_1 \end{cases}$$

and the inequality (2.6) is also proved. ∎

3. Some Bounds in Terms of Δ−Seminorms

The following result of Grüss type holds.

Theorem 2. *Let $f, g : [a,b] \to \mathbb{R}$ be measurable on $[a,b]$. Then we have the inequality:*

$$(3.1)\qquad |T(f,g;a,b)| \le \frac{1}{2(b-a)^2} \|f\|_p^\Delta \|g\|_q^\Delta,$$

where $p = 1$, $q = \infty$, or $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ or $q = 1$ and $p = \infty$, provided all integrals involved exist. Further, $T(f,g;a,b)$ is the Chebychev functional defined by (1.1).

Proof. Using Korkine's identity, we have

$$T(f,g;a,b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) \, dxdy.$$

Now, if $f \in L_\infty [a,b]$, then

$$\begin{aligned} & |T(f,g;a,b)| \\ \leq & \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(x) - f(y)| \, |g(x) - g(y)| \, dxdy \\ \leq & \frac{1}{2(b-a)^2} \, ess \sup_{(x,y)\in[a,b]^2} (f(x) - f(y)) \int_a^b \int_a^b |g(x) - g(y)| \, dxdy \\ = & \frac{1}{2(b-a)^2} \|f\|_\infty^\Delta \|g\|_1^\Delta , \end{aligned}$$

and the inequality is proved for $p = \infty$, $q = 1$.

A similar argument applies for $p = 1$, $q = \infty$.

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then applying Hölder's integral inequality for double integrals, we deduce that

$$\begin{aligned} & |T(f,g;a,b)| \\ \leq & \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(x) - f(y)| \, |g(x) - g(y)| \, dxdy \\ \leq & \frac{1}{2(b-a)^2} \left(\int_a^b \int_a^b |f(x) - f(y)|^p \, dxdy \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |g(x) - g(y)|^q \, dxdy \right)^{\frac{1}{q}} \\ \leq & \frac{1}{2(b-a)^2} \|f\|_p^\Delta \|g\|_q^\Delta \end{aligned}$$

and the theorem is proved. ∎

Remark 1. *Taking into account by Theorem 2 that for* $p = 1$*, we have three bounds for* $\|f\|_1^\Delta$ *and for* $p \in (1,\infty)$ *we have another three bounds for* $\|f\|_p^\Delta$ *and for* $p = \infty$*, we can state some other three bounds by* $\|f\|_\infty^\Delta$*, then, by the inequality (3.1), we are able to point out eighty-one bounds for the modulus of the functional* $T(f,g;a,b)$*, in terms of the derivatives* f' *and* g'*.*

In some practical applications, the $\Delta-$seminorm of a mapping, say f, can be easily computed. In that case, the number of bounds is much less.

The following result for the trapezoid formula holds.

Theorem 3. *Assume that the mapping* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$*. Then we have the inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \tag{3.2}$$

$$\leq B := \begin{cases} \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}-1}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_q^\Delta & \textit{if } \; p \in [1,\infty) \textit{ and } f' \in L_q [a,b] \, ; \; \frac{1}{p} + \frac{1}{q} = 1 \\ \textit{(for } p = 1 \textit{ we choose } q = \infty \textit{);} & \\ \frac{1}{2(b-a)} \|f'\|_1^\Delta . & \end{cases}$$

Proof. We know the following identity (see [12]) holds, where many other related results are given,

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)\,dt, \tag{3.3}$$

which can be easily proved by applying the integration by parts formula.

We observe that

$$T\left(\cdot-\frac{a+b}{2},f',a,b\right)=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)\,dt.$$

If we define $h(t):=t-\frac{a+b}{2}$, and

$$D_p(a,b):=\int_a^b\int_a^b|x-y|^p\,dxdy=2\frac{(b-a)^{p+2}}{(p+1)(p+2)}, \tag{3.4}$$

then we observe that for $p\geq 1$, from (2.9) and (2.10),

$$\|h\|_p^{\Delta}=D_p^{\frac{1}{p}}(a,b)=\frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}$$

and

$$\|h\|_\infty^{\Delta}=ess\sup_{(x,y)\in[a,b]^2}|x-y|=b-a$$

for which, using (3.1), we conclude the desired inequality (3.2). ∎

Corollary 1. *With the assumptions of Theorem 3 and if $f'\in L_2[a,b]$, then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2\sqrt{3}}\left[(b-a)\|f'\|_2^2-[f(b)-f(a)]^2\right]^{\frac{1}{2}}. \tag{3.5}$$

The proof follows by (3.2) for $p=q=2$.

For a different proof, see [14].

Remark 2. *If we take*

$$H(t)=t-z,\quad z\in[a,b],$$

then we would obtain

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left(\frac{z-a}{b-a}f(a)+\frac{b-z}{b-a}f(b)\right)+2\left(\frac{a+b}{2}-z\right)\left(\frac{f(b)-f(a)}{b-a}\right)\right|\leq B,$$

where the bound B is as defined in (3.2) and is independent of z. If $z=\frac{a+b}{2}$, then the perturbation resulting from the application of the Grüss identity vanishes and the results of Theorem 3 are recaptured.

The following result for the midpoint formula holds.

Theorem 4. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.6}$$

$$\leq \quad B:=\begin{cases}\frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}-1}}{[(p+1)(p+2)]^{\frac{1}{p}}}\left\|f'\right\|_q^{\Delta} & \text{if } p\in[1,\infty) \text{ and } f'\in L_q[a,b]\,;\\ & \frac{1}{p}+\frac{1}{q}=1, \text{ (for } p=1 \text{ we choose } q=\infty);\\ \frac{1}{2(b-a)}\left\|f'\right\|_1^{\Delta}. & \end{cases}$$

Proof. A simple integration by parts demonstrates that the following identity holds:

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b k(t)\,f'(t)\,dt, \tag{3.7}$$

where

$$k(t)=\begin{cases} t-a & \text{if} \quad t\in\left[a,\frac{a+b}{2}\right]\\ t-b & \text{if} \quad t\in\left(\frac{a+b}{2},b\right],\end{cases}$$

which can easily be proved using the integration by parts formula.

We observe that

$$T(k,f';a,b)=\frac{1}{b-a}\int_a^b k(t)\,f'(t)\,dt,$$

as a simple computation shows that

$$\frac{1}{b-a}\int_a^b k(t)\,dt=0.$$

We observe that

$$\|k\|_\infty^{\Delta}=ess\sup_{(x,y)\in[a,b]^2}|k(x)-k(y)|=b-a.$$

Also, we have:

$$\begin{aligned}\|k\|_p^{\Delta} &= \left(\int_a^b\int_a^b |k(x)-k(y)|^p\,dxdy\right)^{\frac{1}{p}}\\ &= \left[\int_a^b\left(\int_a^{\frac{a+b}{2}}|k(x)-y+a|^p\,dy+\int_{\frac{a+b}{2}}^b|k(x)-y+b|^p\,dy\right)dx\right]^{\frac{1}{p}}\\ &= \left[\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}}|x-y|^p\,dy\right)dx+\int_{\frac{a+b}{2}}^b\left(\int_a^{\frac{a+b}{2}}|x-b-y+a|^p\,dy\right)dx\right.\\ &\quad\left.+\int_a^{\frac{a+b}{2}}\left(\int_{\frac{a+b}{2}}^b|x-a-y+b|^p\,dy\right)dx+\int_{\frac{a+b}{2}}^b\left(\int_{\frac{a+b}{2}}^b|x-y|^p\,dy\right)dx\right]^{\frac{1}{p}}\\ &: = I_1+I_2+I_3+I_4.\end{aligned}$$

We have

$$I_1 = \int_a^{\frac{a+b}{2}} \left(\int_a^{\frac{a+b}{2}} |x-y|^p \, dy \right) dx = D_p\left(a, \frac{a+b}{2}\right)$$

and so, from (3.4),

$$I_1 = \frac{2\left(\frac{b-a}{2}\right)^{p+2}}{(p+1)(p+2)} = \frac{(b-a)^{p+2}}{2^{p+1}(p+1)(p+2)} := \frac{D_p(a,b)}{2^{p+1}}.$$

Further,

$$\begin{aligned}
I_2 &= \int_{\frac{a+b}{2}}^b \left(\int_a^{\frac{a+b}{2}} |x-(y+b-a)|^p \, dy \right) dx \\
&= \int_{\frac{a+b}{2}}^b \left(\int_b^{b+\frac{b-a}{2}} |x-u|^p \, du \right) dx = \int_{\frac{a+b}{2}}^b \left(\int_b^{b+\frac{b-a}{2}} (u-x)^p \, du \right) dx \\
&= \int_{\frac{a+b}{2}}^b \left(\left. \frac{(u-x)^{p+1}}{p+1} \right|_b^{b+\frac{b-a}{2}} \right) dx \\
&= \int_{\frac{a+b}{2}}^b \left[\frac{\left(b+\frac{b-a}{2}-x\right)^{p+1} - (b-x)^{p+1}}{p+1} \right] dx \\
&= \frac{(b-a)^{p+2}}{(p+1)(p+2)} - \frac{(b-a)^{p+2}}{2^{p+1}(p+1)(p+2)} = \left(1 - \frac{1}{2^{p+1}}\right) D_p(a,b).
\end{aligned}$$

Now,

$$I_3 = \int_a^{\frac{a+b}{2}} \left(\int_{\frac{a+b}{2}}^b |x-(y+a-b)|^p \, dy \right) dx$$

and following a similar argument to the calculation of I_2 gives

$$I_3 = \left(1 - \frac{1}{2^{p+1}}\right) D_p(a,b).$$

An alternate approach is that a substitution of $Y = y - \frac{b-a}{2}$ and $X = x + \frac{b-a}{2}$ in I_3 shows that $I_3 = I_2$.

Now, from (3.4),

$$\begin{aligned}
I_4 &= \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^b |x-y|^p \, dy \right) dx = D_p\left(\frac{a+b}{2}, b\right) \\
&= D_p\left(a, \frac{a+b}{2}\right) = \frac{D_p(a,b)}{2^{p+1}}.
\end{aligned}$$

Consequently,

$$I = I_1 + I_2 + I_3 + I_4 = 2D_p(a,b) = \frac{2(b-a)^{p+2}}{(p+1)(p+2)}$$

and so

$$\|k\|_p^{\Delta} = \frac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}.$$

Using Theorem 2, we obtain the desired inequality (2.6). ∎

Corollary 2. *With the assumptions of Theorem 4 and if $f' \in L_2[a,b]$, we have the inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2\sqrt{3}} \left[(b-a)\|f'\|_2^2 - [f(b)-f(a)]^2 \right]^{\frac{1}{2}}. \tag{3.8}$$

The proof follows by Theorem 4 applied for $p = q = 2$.

For a different proof of this inequality see [14].

Remark 3. *If we take*

$$K(t) = \begin{cases} t-a, & t \in [a,z] \\ t-b, & t \in (z,b] \end{cases} \tag{3.9}$$

then the following identity attributed to Montgomery (see [13, p. 565]*) may be easily shown to hold*

$$f(z) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b K(t) f'(t)\,dt. \tag{3.10}$$

Now, from (1.1), (3.9) and (3.10)

$$-T(K, f', a, b) = \frac{1}{b-a}\int_a^b f(t)\,dt - f(z) + \left(z - \frac{a+b}{2}\right)\frac{(f(b)-f(a))}{b-a} \tag{3.11}$$

since

$$\frac{1}{b-a}\int_a^b K(t)\,dt = z - \frac{a+b}{2} \quad \text{and} \quad \frac{1}{b-a}\int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a}.$$

We note that from (3.9)

$$\|K\|_\infty^{\Delta} = ess \sup_{(x,y)\in[a,b]^2} |K(x) - K(y)| = b-a$$

and for $p \geq 1$

$$\begin{aligned} \|K\|_p^{\Delta} &= \left(\int_a^b \int_a^b |K(x)-K(y)|^p\,dydx\right)^{\frac{1}{p}} \\ &= \left\{\int_a^z \int_a^z |x-y|^p\,dydx + \int_z^b \int_a^z |x-b-(y-a)|^p\,dydx \right. \\ &\quad \left. + \int_a^z \int_z^b |x-a-(y-b)|^p\,dydx + \int_z^b \int_z^b |x-y|^p\,dydx \right\}^{\frac{1}{p}} \\ &:= (J_1 + J_2 + J_3 + J_4)^{\frac{1}{p}}. \end{aligned} \tag{3.12}$$

Now, from (3.3)

$$J_1 = D_p(a,z) = \frac{2(z-a)^{p+2}}{(p+1)(p+2)}$$

and

$$J_4 = D_p(z,b) = \frac{2(b-z)^{p+2}}{(p+1)(p+2)}.$$

Further,

$$\begin{aligned} J_2 &= \int_z^b \int_a^z |x-b-(y-a)|^p \, dydx = \int_z^b \int_b^{b+z-a} |x-u|^p \, dudx \\ &= \int_z^b \int_b^{b+z-a} (u-x)^p \, dudx = \frac{1}{p+1} \int_z^b (b+z-a-x)^{p+1} - (b-x)^{p+1} \, dx \\ &= \frac{1}{(p+1)(p+2)} \left[(b-a)^{p+2} - (z-a)^{p+2} - (b-z)^{p+2} \right] \\ &= D_p(a,b) - D_p(a,z) - D_p(z,b). \end{aligned}$$

Using symmetry arguments or direct calculation shows that $J_3 = J_2$*. Hence, from (3.12)*

$$\|K\|_p^{\Delta} = 2D_p(a,b) = \frac{2(b-a)^{p+2}}{(p+1)(p+2)}$$

and so, from (3.11)

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(z) + \left(z - \frac{a+b}{2}\right) \left(\frac{f(b)-f(a)}{b-a} \right) \right| \leq B,$$

giving the same bounds as obtained previously for the trapezoidal and midpoint rules. If $z = \frac{a+b}{2}$*, then the midpoint rule is recaptured.*

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