ON SOME INTEGRAL INEQUALITIES INVOLVING CONVEX FUNCTIONS

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Abstract. In this paper we establish some new integral inequalities involving convex functions by using a fairly elementary analysis.

1. Introduction

Let $f, g : [a, b] \to \mathbb{R}$ be convex mappings. For two elements x, y in $[a, b]$, we shall define the mappings $F(x, y)$, $G(x, y) : [0, 1] \to \mathbb{R}$ as follows

$$F(x,y)(t) = \frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right], \tag{1.1}$$

$$G(x,y)(t) = \frac{1}{2}\left[g(tx + (1-t)y) + g((1-t)x + ty)\right]. \tag{1.2}$$

Recently in [2] Dragomir and Ionescu established some interesting properties of such mappings. In particular, in [2], it is shown that $F(x, y)$, $G(x, y)$ are convex on $[0, 1]$. In another paper [6], Pečarić and Dragomir proved that the following statements are equivalent for mappings $f, g : [a, b] \to \mathbb{R}$:

(i) f, g are convex on $[a, b]$;
(ii) for all $x, y \in [a, b]$ the mappings f_0, $g_0 : [0, 1] \to \mathbb{R}$ defined by $f_0(t) = f(tx + (1-t)y)$ or $f((1-t)x + ty)$, $g_0(t) = g(tx + (1-t)y)$ or $g((1-t)x + ty)$ are convex on $[0, 1]$.

From these properties, it is easy to observe that if f_0 and g_0 are convex on $[0, 1]$, they are integrable on $[0, 1]$ and hence f_0g_0 is also integrable on $[0, 1]$. Similarly, if f and g are convex on $[a, b]$, they are integrable on $[a, b]$ and hence fg is also integrable on $[a, b]$. Consequently, it is easy to see that if f and g are convex on $[a, b]$, then $F = F(x, y)$ and $G = G(x, y)$ and hence Fg Gf, Ff, Gg are also integrable on $[a, b]$. We shall use these facts in our discussion without further mention.

The object of this paper is to establish some new integral inequalities involving the functions F and G as defined in (1.1) and (1.2). The analysis used in the proof is elementary and we believe that the inequalities established here are of independent interest. For other results related to such inequalities, see [1] - [6] where further references are given.

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2. Statement of Results

Our main result is given in the following theorem.

Theorem 1. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$ and the mappings $F(x,y)$ and $G(x,y)$ be defined by (1.1) and (1.2). Then for all t in $[0,1]$ we have*

$$\frac{1}{(b-a)^2}\int_a^b (b-y) f(y) g(y)\,dy \leq \frac{2}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_a^y [F(x,y)(t)g(x)+G(x,y)(t)f(x)]\,dx\right)dy + \frac{1}{10}f(a)g(a), \tag{2.1}$$

$$\frac{1}{(b-a)^2}\int_a^b (y-a) f(y) g(y)\,dy \leq \frac{2}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_y^b [F(x,y)(t)g(x)+G(x,y)(t)f(x)]\,dx\right)dy + \frac{1}{10}f(b)g(b), \tag{2.2}$$

$$\frac{1}{(b-a)}\int_a^b f(y) g(y)\,dy \leq \frac{2}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b [F(x,y)(t)g(x)+G(x,y)(t)f(x)]\,dxdy + \frac{1}{10}[f(a)g(a)+f(b)g(b)]. \tag{2.3}$$

Our next result deals with the slight variants of the inequalities given in Theorem 1.

Theorem 2. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$ and the mappings $F(x,y)$ and $G(x,y)$ be defined by (1.1) and (1.2). Then for all t in $[0,1]$ we have*

$$\frac{1}{(b-a)^2}\int_a^b (b-y)\left[f^2(y)+g^2(y)\right]dy \leq \frac{4}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_a^y [F(x,y)(t)f(x)+G(x,y)(t)g(x)]\,dx\right)dy + \frac{1}{10}\left[f^2(a)+g^2(a)\right], \tag{2.4}$$

$$\frac{1}{(b-a)^2}\int_a^b (y-a)\left[f^2(y)+g^2(y)\right]dy \leq \frac{4}{5}\cdot\frac{1}{(b-a)^2}\int_a^b\left(\int_y^b [F(x,y)(t)f(x)+G(x,y)(t)g(x)]\,dx\right)dy + \frac{1}{10}\left[f^2(b)+g^2(b)\right], \tag{2.5}$$

$$
\begin{aligned}
(2.6) \qquad & \frac{1}{(b-a)} \int_a^b \left[f^2(y)+g^2(y)\right] dy \\
\leq \; & \frac{4}{5} \cdot \frac{1}{(b-a)^2} \int_a^b \int_a^b \left[F(x,y)(t) f(x) + G(x,y)(t) g(x)\right] dxdy \\
& + \frac{1}{10} \left[f^2(a)+g^2(a)+f^2(b)+g^2(b)\right].
\end{aligned}
$$

3. Proof of Theorem 1

Proof. The assumptions that f and g are nonnegative and convex imply that we may assume that $f,\, g \in C^1$ and that we have the following estimates

$$f(tx+(1-t)y) \geq f(x)+(1-t)(y-x) f'(x), \tag{3.1}$$

$$f((1-t)x+ty) \geq f(x)+t(y-x) f'(x), \tag{3.2}$$

$$g(tx+(1-t)y) \geq g(x)+(1-t)(y-x) g'(x), \tag{3.3}$$

$$g((1-t)x+ty) \geq g(x)+t(y-x) g'(x), \tag{3.4}$$

for $x, y \in [a,b]$ and $t \in [0,1]$. From (3.1), (3.2) (1.1) and (3.3), (3.4), (1.2) it is easy to see that

$$F(x,y)(t) \geq f(x) + \frac{1}{2}(y-x) f'(x), \tag{3.5}$$

$$G(x,y)(t) \geq g(x) + \frac{1}{2}(y-x) g'(x), \tag{3.6}$$

for $x, y \in [a,b]$ and $t \in [0,1]$. Multiplying (3.5) by $g(x)$ and (3.6) by $f(x)$ and then adding, we obtain

$$
\begin{aligned}
(3.7) \qquad & F(x,y)(t) g(x) + G(x,y)(t) f(x) \\
\geq \; & 2f(x) g(x) + \frac{1}{2}(y-x) \frac{d}{dx}(f(x) g(x)).
\end{aligned}
$$

Integrating the inequality (3.7) over x from a to y we have

$$
\begin{aligned}
(3.8) \qquad & \int_a^y \left[F(x,y)(t) g(x) + G(x,y)(t) f(x)\right] dx \\
\geq \; & \frac{5}{2} \int_a^y f(x) g(x) dx - \frac{1}{2}(y-a) f(a) g(a).
\end{aligned}
$$

Further, integrating both sides of (3.8) with respect to y from a to b we get

$$
\begin{aligned}
(3.9) \qquad & \int_a^b \int_a^y \left[F(x,y)(t) g(x) + G(x,y)(t) f(x)\right] dxdy \\
\geq \; & \frac{5}{2} \int_a^b (b-y) f(y) g(y) dy - \frac{1}{4}(b-a)^2 f(a) g(a).
\end{aligned}
$$

Multiplying both sides of (3.9) by $\frac{2}{5} \cdot \frac{1}{(b-a)^2}$ and rewriting we get the required inequality in (2.1).

Similarly, by first integrating (3.7) over x from y to b and then integrating the resulting inequality over y from a to b, we get the required inequality in (2.2). The inequality (2.3) is obtained by adding the inequalities (2.1) and (2.2). The proof is complete. ∎

4. Proof of Theorem 2

Proof. As in the proof of Theorem 1, from the assumptions we have the estimates (3.5) and (3.6). Multiplying (3.5) by $f(x)$ and (3.6) by $g(x)$ and then adding, we obtain

$$\begin{aligned} & F(x,y)(t) f(x) + G(x,y)(t) g(x) \\ \geq\ & f^2(x) + g^2(x) + \frac{1}{2}(y-x)\left[f(x) f'(x) + g(x) g'(x)\right]. \end{aligned} \tag{4.1}$$

Integrating (4.1) over x from a to y, we have

$$\begin{aligned} & \int_a^y \left[F(x,y)(t) f(x) + G(x,y)(t) g(x)\right] dx \\ \geq\ & \frac{5}{4}\int_a^y \left[f^2(x) + g^2(x)\right] dx - \frac{1}{4}(y-a)\left[f^2(a) + g^2(a)\right]. \end{aligned} \tag{4.2}$$

Further, integrating both sides of (4.2) with respect to y from a to b we have

$$\begin{aligned} & \int_a^b \left(\int_a^y \left[F(x,y)(t) f(x) + G(x,y)(t) g(x)\right] dx\right) dy \\ \geq\ & \frac{5}{4}\int_a^b (b-y)\left[f^2(y) + g^2(y)\right] dy - \frac{1}{8}(b-a)^2\left[f^2(a) + g^2(a)\right]. \end{aligned} \tag{4.3}$$

Multiplying both sides of (4.3) by $\frac{4}{5}\cdot\frac{1}{(b-a)^2}$ and rewriting, we get the required inequality in (2.4).

The remainder of the proof follows by the same arguments as mentioned in the proof of Theorem 1 with suitable modifications and hence the proof is complete. ∎

5. Further Inequalities

In this section we shall give some inequalities that are analogous to those given in Theorem 1 involving only one convex function. We believe that these inequalities are interesting in their own right.

Theorem 3. *Let f be a real-valued nonnegative convex function on $[a,b]$. Then*

$$\begin{aligned} & \frac{1}{(b-a)^2}\int_a^b (b-y) f(y)\, dy \\ \leq\ & \frac{2}{3}\cdot\frac{1}{(b-a)^2}\int_a^b \left[\int_a^y \left(\int_0^1 f(tx + (1-t)y)\, dt\right) dx\right] dy + \frac{1}{6} f(a), \end{aligned} \tag{5.1}$$

$$\begin{aligned} & \frac{1}{(b-a)^2}\int_a^b (y-a) f(y)\, dy \\ \leq\ & \frac{2}{3}\cdot\frac{1}{(b-a)^2}\int_a^b \left[\int_y^b \left(\int_0^1 f(tx + (1-t)y)\, dt\right) dx\right] dy + \frac{1}{6} f(b), \end{aligned} \tag{5.2}$$

$$\begin{aligned}(5.3)\qquad & \frac{1}{(b-a)}\int_a^b f(y)\,dy \\ \leq\quad & \frac{2}{3}\cdot\frac{1}{(b-a)^2}\int_a^b\left[\int_a^b\left(\int_0^1 f(tx+(1-t)y)\,dt\right)dx\right]dy \\ & +\frac{1}{6}\left[f(a)+f(b)\right].\end{aligned}$$

Proof. To prove the inequality (5.1), as in the proof of Theorem 1 from the assumptions we have the estimate (3.1). Integrating both sides of (3.1) over t from 0 to 1 we have

$$\int_0^1 f(tx+(1-t)y)\,dt \geq f(x)+\frac{1}{2}(y-x)f'(x). \tag{5.4}$$

Now first integrating both sides of (5.4) over x from a to y and after that integrating the resulting inequality over y from a to b we get the required inequality in (5.1).

Similarly, by first integrating both sides of (5.4) over x from y to b and then integrating the resulting inequality over y from a to b we get the inequality in (5.2). By adding the inequalities (5.1) and (5.2) we get the inequality (5.3). The proof of Theorem 3 is thus completed. ∎

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