ESTIMATES FOR AN INTEGRAL IN L^p NORM OF THE $(n+1)$-TH DERIVATIVE OF ITS INTEGRAND

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ABSTRACT. Basing on Taylor's formula with an integral remaider, an integral is estimated in L^p norm of the $(n+1)$-th derivative of its integrand, and the Iyengar's inequality and many other useful inequalities are generalized.

1. Introduction

The study of Iyengar's inequality [6] has a rich literature and a long history. For details please refer to references in this article.

In [4, 5] and [11]–[15], making use of mean value theorems for derivative (including Rolle's, Lagarange's, and Taylor's), mean value theorem for integral, the Taylor's formula for functions of several variables, and other technique, the following result was obtained:

Theorem 1. *Let the function $f \in C^n([a,b])$ have derivative of $(n+1)$-th order in (a,b) satisfying $N \leqslant f^{(n+1)}(x) \leqslant M$. Denote*

$$
\begin{aligned}
S_n(u,v,w) &= \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \cdot u^k f^{(k-1)}(v) + \frac{w}{n!} \cdot (-1)^n u^n,\\
\frac{\partial^k S_n}{\partial u^k} &= S_n^{(k)}(u,v,w),
\end{aligned} \tag{1}
$$

then, for any $t \in (a,b)$, when n is odd we have

$$
\begin{aligned}
\sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a,a,N) - S_{n+2}^{(i)}(b,b,N)\right) t^i &\leqslant \int_a^b f(x)\,\mathrm{d}x\\
&\leqslant \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a,a,M) - S_{n+2}^{(i)}(b,b,M)\right) t^i;
\end{aligned} \tag{2}
$$

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when n is even we have

$$\sum_{i=0}^{n+2}\frac{(-1)^i}{i!}\left(S_{n+2}^{(i)}(a,a,N)-S_{n+2}^{(i)}(b,b,M)\right)t^i\leqslant\int_a^b f(x)\,\mathrm{d}x \leqslant\sum_{i=0}^{n+2}\frac{(-1)^i}{i!}\left(S_{n+2}^{(i)}(a,a,M)-S_{n+2}^{(i)}(b,b,N)\right)t^i. \tag{3}$$

In [1]–[3], [8] and [18], similar problems were investigated by using Hayashi's inequality. For more information, please see [7], [9] and [10].

In this article, by the Taylor's mean value theorem [17] with an integral remainder, we have

Theorem 2. *Let the function $f\in C^n([a,b])$ have derivative of $(n+1)$-th order in (a,b), and $f^{(n+1)}\in L^p([a,b])$ for positive numbers p and q satisfying $\frac1p+\frac1q=1$. Then, for any $t\in(a,b)$, we have*

$$\left|\int_a^b f(x)\,\mathrm{d}x-\sum_{i=0}^{n}\frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1}+\sum_{i=0}^{n}\frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}\right| \leqslant\frac{(t-a)^{n+1+\frac1q}+(b-t)^{n+1+\frac1q}}{(n+1)!\sqrt[q]{nq+q+1}}\left\|f^{(n+1)}\right\|_{L^p([a,b])}. \tag{4}$$

Corollary 1. *Let $f\in C^n([a,b])$. If $f^{(i)}(a)=f^{(i)}(b)=0$ for $1\leqslant i\leqslant n$, and $f^{(n+1)}\in L^p([a,b])$ is not identically zero, then for positive numbers p and q satisfying $\frac1p+\frac1q=1$, we have*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right|\leqslant\frac{(b-a)^{1+\frac1q}}{\sqrt[q]{2(q+1)}}\left\|f^{(n+1)}\right\|_{L^p([a,b])}. \tag{5}$$

2. Proof of Theorem 2

Let t be a parameter satisfying $a<t<b$, and write

$$\int_a^b f(x)\,\mathrm{d}x=\int_a^t f(x)\,\mathrm{d}x+\int_t^b f(x)\,\mathrm{d}x. \tag{6}$$

The well-known Taylor's formula with an integral remainder [17, pp. 4–6] states that

$$f(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i+\frac{1}{n!}\int_a^x(x-s)^nf^{(n+1)}(s)\,\mathrm{d}s, \tag{7}$$

$$f(x)=\sum_{i=0}^{n}\frac{f^{(i)}(b)}{i!}(x-b)^i+\frac{1}{n!}\int_b^x(x-s)^nf^{(n+1)}(s)\,\mathrm{d}s, \tag{8}$$

and

$$\int_a^x\int_a^t(t-s)^nf^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}t=\frac{1}{n+1}\int_a^x(x-s)^{n+1}f^{(n+1)}(s)\,\mathrm{d}s. \tag{9}$$

Integrating on both sides of (7) over $[a,t]$, we obtain

$$\begin{aligned}\int_a^t f(x)\,\mathrm{d}x &= \sum_{i=0}^{n}\frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1}+\frac{1}{n!}\int_a^t\int_a^x (x-s)^n f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x\\ &= \sum_{i=0}^{n}\frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1}+\frac{1}{(n+1)!}\int_a^t (t-s)^{n+1} f^{(n+1)}(s)\,\mathrm{d}s.\end{aligned}\tag{10}$$

Therefore, we have

$$\begin{aligned}&\left|\int_a^t f(x)\,\mathrm{d}x-\sum_{i=0}^{n}\frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1}\right|\\ \leqslant&\frac{1}{(n+1)!}\int_a^t\left|(t-s)^{n+1}f^{(n+1)}(s)\right|\,\mathrm{d}s\\ \leqslant&\frac{1}{(n+1)!}\left(\int_a^t (t-s)^{q(n+1)}\,\mathrm{d}s\right)^{1/q}\left(\int_a^t\left|f^{(n+1)}(s)\right|^p\,\mathrm{d}s\right)^{1/p}\\ \leqslant&\frac{1}{(n+1)!}\left(\frac{(t-a)^{nq+q+1}}{nq+q+1}\right)^{1/q}\left\|f^{(n+1)}\right\|_{L^p[a,b]},\end{aligned}\tag{11}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Integrating on both sides of (8) over $[t,b]$, we get

$$\begin{aligned}\int_t^b f(x)\,\mathrm{d}x &= -\sum_{i=0}^{n}\frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}+\frac{1}{n!}\int_t^b\int_b^x (x-s)^n f^{(n+1)}(s)\,\mathrm{d}s\,\mathrm{d}x\\ &= -\sum_{i=0}^{n}\frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}+\frac{1}{(n+1)!}\int_t^b (t-s)^{n+1} f^{(n+1)}(s)\,\mathrm{d}s.\end{aligned}\tag{12}$$

Thus, from (12), it follows that

$$\begin{aligned}&\left|\int_t^b f(x)\,\mathrm{d}x+\sum_{i=0}^{n}\frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}\right|\\ \leqslant&\frac{1}{(n+1)!}\int_t^b\left|(t-s)^{n+1}f^{(n+1)}(s)\right|\,\mathrm{d}s\\ \leqslant&\frac{1}{(n+1)!}\left(\int_t^b (s-t)^{p(n+1)}\,\mathrm{d}s\right)^{1/p}\left(\int_t^b\left|f^{(n+1)}(s)\right|^p\,\mathrm{d}s\right)^{1/p}\\ \leqslant&\frac{1}{(n+1)!}\left(\frac{(b-t)^{np+p+1}}{np+p+1}\right)^{1/p}\left\|f^{(n+1)}\right\|_{L^p[a,b]}.\end{aligned}\tag{13}$$

From (6), (11) and (13), we have

$$\begin{aligned}
&\left|\int_a^b f(x)\,\mathrm{d}x-\sum_{i=0}^{n}\frac{f^{(i)}(a)}{(i+1)!}(t-a)^{i+1}+\sum_{i=0}^{n}\frac{f^{(i)}(b)}{(i+1)!}(t-b)^{i+1}\right|\\
&\leqslant\frac{1}{(n+1)!}\left\{\left(\frac{(t-a)^{np+p+1}}{np+p+1}\right)^{1/p}+\left(\frac{(b-t)^{np+p+1}}{np+p+1}\right)^{1/p}\right\}\cdot\left\|f^{(n+1)}\right\|_{L^p[a,b]}\\
&=\frac{(t-a)^{n+1+\frac{1}{q}}+(b-t)^{n+1+\frac{1}{q}}}{(n+1)!\sqrt[q]{nq+q+1}}\left\|f^{(n+1)}\right\|_{L^p([a,b])}.
\end{aligned}\tag{14}$$

The proof is complete.

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