EXPONENTIAL STABILITY FOR PERIODIC EVOLUTION FAMILIES OF BOUNDED LINEAR OPERATORS

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Abstract. We prove that a q-periodic evolution family

$$\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$$

of bounded linear operators is uniformly exponentially stable if and only if

$$\sup_{t>0} || \int_0^t e^{-i\mu\xi} U(t,\xi) f(\xi) d\xi || = M(\mu, f) < \infty$$

for all $\mu \in \mathbf{R}$ and $f \in P_q(\mathbf{R}_+, X)$, (that is f is a q-periodic and continuous function on $\mathbf{R}_+$).

1. Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear operators acting on X. We denote by $||\cdot||$, the norms of vectors and operators. Let $A \in \mathcal{L}(X)$ and $\mathbf{R}_+$, the set of the all non-negative real numbers. It is known, see e.g. [1] that if the Cauchy Problem

$$\dot{x}(t) = Ax(t) + e^{i\mu t} x_0, \quad x(0) = 0,$$

has a bounded solution on $\mathbf{R}_+$ for every $\mu \in \mathbf{R}$ and any $x_0 \in X$ then the homogenous system $\dot{x} = Ax$, is uniformly exponentially stable. The hypothesis of the above result can be write in the form:

$$\sup_{t>0} || \int_0^t e^{-i\mu\xi} e^{\xi A} x_0 d\xi || < \infty, \quad \forall \mu \in \mathbf{R}, \forall x_0 \in X.$$

This result cannot be extended for C_0-semigroups, cf. [14, Example 3.1]. However, Neerven in [11, Corollary 5] shown that if $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X and

$$\sup_{\mu\in\mathbf{R}} \sup_{t>0} || \int_0^t e^{i\mu\xi} T(\xi) x_0 d\xi || < \infty, \quad \forall x_0 \in X, \tag{1}$$

then $\omega_1(\mathbf{T}) < 0$. For details concerning $\omega_1(\mathbf{T})$, we refer to [12] or [9, Theorem A IV.1.4]. Moreover, under the hypothesis (1), it results that the resolvent $R(z, A_{\mathbf{T}}) = (z - A_{\mathbf{T}})^{-1}$ of the infinitesimal generator of $\mathbf{T}$, exists and is uniformly bounded on $\mathbf{C}_+ := \{\lambda \in \mathbf{C} : \text{ Re } (\lambda) > 0\}$, see [11]. Combining this with a result of Gearhart [6], (see also Huang [7], Weiss [15] or Pandolfi [13] for other proofs and generalizations), it results that if X is a complex Hilbert space and (1) holds, then $\mathbf{T}$ is uniformly exponentially stable, i.e. its growth bound $\omega_0(\mathbf{T})$ is negative. A similar problem for q-evolution families of bounded linear operators seems to be an open question.

In the general case, when X is a Banach space the last results is not true, see e.g. [2, Example 2]. However, a weakly result, announced before, holds.

2. Definitions. Preliminary results

Let $q > 0$ and $\Delta = \{(t,s) \in \mathbf{R}^2 : t \geq s \geq 0\}$. A mapping $\mathcal{U} : \Delta \to \mathcal{L}(X)$ would be called q-periodic evolution family of bounded linear operators on X, iff:

- **(i)** $U(t,s) = U(t,r)U(r,s)$ for all $t \geq s \geq r \geq 0$;
- **(ii)** $U(t,t) = Id$, (Id is the identity on X), for all $t \geq 0$;
- **(iii)** for all $x \in X$, the map $(t,s) \mapsto U(t,s)x : \Delta \to X$, is continuous;
- **(iv)** $U(t+q, s+q) = U(t,s)$ for all $t \geq s \geq 0$.

The operator $\mathcal{U}(t,s)$ was denoted by $U(t,s)$.

If A is a linear operator on X, $\sigma(A)$ will denote the *spectrum of* A, and if $T \in \mathcal{L}(X)$, $r(T)$ will denote the *spectral radius* of T.

The following two lemmas, which would be used later, are essentially known (see [4, Ch.V, Theorem 1.1, Corollary 1.1] or [5, Theorem 6.6]).

LEMMA 1. *A q-periodic evolution family $\mathcal{U}$ on X has exponential growth, that is, there are $\omega \in \mathbf{R}$ and $M > 1$ such that*

$$||U(t,s)|| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0. \tag{2}$$

We recall that the evolution family $\mathcal{U}$ is called *exponentially stable* if there are $\omega < 0$ and $M > 1$ such that (2) holds. Let $V = U(q,0) \in \mathcal{L}(X)$.

LEMMA 2. *A q-periodic evolution family $\mathcal{U}$ is exponentially stable if and only if $r(V) < 1$.*

For the proofs of these lemmas we refer to [3].

Let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on X and $A_{\mathbf{T}}$ its infinitesimal generator. In [14, Proposition 3.3] it is been shown that if

$$\sup_{t>0} || \int_0^t e^{i\mu\xi} T(\xi) d\xi || < \infty, \quad \forall x \in X, \forall \mu \in \mathbf{R}$$

then

$$\sigma(A_{\mathbf{T}}) \subset \mathbf{C}_- := \{z \in \mathbf{C} : \text{ Re } (z) < 0\}.$$

The discret version of this result is the following:

LEMMA 3. *Let $T \in \mathcal{L}(X)$. If*

$$\sup_{n \in \mathbf{N}} || \sum_{k=0}^{n} e^{i\mu k} T^k || = M_\mu < \infty \quad \forall \mu \in \mathbf{R},$$

then $r(T) < 1$.

We mention that the result in Lemma 3 is also known and is, for instance, consequence of the uniform ergodic theorem [8, Theorem 2.1, Theorem 2.7]. For reasons of self-containedness we give the proof of Lemma 3 in detail.

Proof. We will use the identity:

$$\sum_{k=0}^{n} e^{i\mu k}T^k(e^{i\mu}T - Id) = e^{i\mu(n+1)}T^{n+1} - Id. \tag{3}$$

From (3) it follows:

$$||e^{i\mu(n+1)}T^{n+1}|| \leq 1 + M_\mu(1 + ||T||) \quad \forall n \in \mathbf{N}, \tag{4}$$

that is $r(T) \leq 1$. Suppose that $1 \in \sigma(T)$. Then for all $m = 1, 2, \cdots$, there exists $x_m \in X$ with $||x_m|| = 1$ and $(Id - T)x_m \to 0$ as $m \to \infty$, (see [9, Proposition 2.2, p. 64]). From (4) it results that $T^k(Id - T)x_m \to 0$ as $m \to \infty$, uniformly for $k \in \mathbf{N}$. Let $N \in \mathbf{N}$, $N > 2M_0$ and $m \in \mathbf{N}$ such that

$$||T^k(Id - T)x_m|| \leq \frac{1}{2N}, \quad k = 0, 1, \cdots N.$$

Then

$$\begin{aligned} M_0 &\geq \|x_m + \sum_{k=1}^{N}(x_m + \sum_{j=0}^{k-1} T^j(T - Id)x_m)\| \\ &= \|(N+1)x_m + \sum_{k=1}^{N}\sum_{j=0}^{k-1} T^j(T - Id)x_m\| \\ &\geq (N+1) - \tfrac{N(N+1)}{4N} > \tfrac{N}{2} > M_0. \end{aligned}$$

This contradiction concludes that $1 \notin \sigma(T)$. Now, it is easy to show that $e^{i\mu} \notin \sigma(T)$ for $\mu \in \mathbf{R}$, that is, $r(T) < 1$.

3. Uniform Exponential Stability

Let us consider the following spaces:

- $BUC(\mathbf{I}, X), \mathbf{I} \in \{\mathbf{R}, \mathbf{R}_+\}$ is the Banach space of all X-valued bounded uniformly continuous functions on $\mathbf{I}$, with sup-norm.
- $AP(\mathbf{I}, X)$ is the linear closed hull in $BUC(\mathbf{I}, X)$ of the set of all functions

$$t \mapsto e^{i\mu t}x : \mathbf{I} \to X, \quad \mu \in \mathbf{R}, \quad x \in X.$$

- $P_q(\mathbf{I}, X)$ is the set of all continuous functions $f : \mathbf{I} \to X$ such that $f(t+q) = f(t)$, for any $t \in \mathbf{I}$ and some $q > 0$.

THEOREM 4. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a q-periodic evolution family on the Banach space X. If*

$$\sup_{t>0} ||\int_0^t e^{-i\mu\xi}U(t,\xi)f(\xi)d\xi|| < \infty, \quad \forall \mu \in \mathbf{R}, \forall f \in P_q(\mathbf{R}_+, X), \tag{5}$$

then $\mathcal{U}$ is exponentially stable.

Proof. Let $V = U(q, 0)$, $x \in X$, $n = 0, 1, \cdots$ and $g \in P_q(\mathbf{R}_+, X)$, such that

$$g(\xi) = \xi(q - \xi)U(\xi, 0)x, \quad \forall \xi \in [0, q].$$

From (5), for $t=(n+1)q$, we obtain:

$$\sup_{n\in\mathbf{N}}\|\sum_{k=0}^{n}\int_{kq}^{(k+1)q}U((n+1)q,\xi)e^{-i\mu\xi}g(\xi)d\xi\|<\infty,\quad\forall\mu\in\mathbf{R}. \tag{6}$$

In the view of definition of q-periodic evolution family **(iv)**, it follows:

$$U(pq+q,pq+u)=U(q,u),\quad\forall p\in\mathbf{N},\quad\forall u\in[0,q]$$

and

$$U(pq,jq)=U((p-j)q,0)=V^{p-j},\quad\forall p\in\mathbf{N},\forall j\in\mathbf{N},\quad p\geq j.$$

Now, for every $k=0,1,\cdots$, we have:

$$\begin{aligned}\int_{kq}^{(k+1)q}U((n+1)q,\xi)e^{-i\mu\xi}g(\xi)d\xi &= \int_{kq}^{(k+1)q}U((n+1)q,(k+1)q)U((k+1)q,\xi)e^{-i\mu\xi}g(\xi)d\xi\\ &= V^{n-k}\int_0^q U((k+1)q,u+kq)e^{-i\mu(u+kq)}g(kq+u)du\\ &= e^{-i\mu kq}V^{n-k}\int_0^q e^{-i\mu u}U(q,u)g(u)du\\ &= e^{-i\mu kq}V^{n-k}\int_0^q e^{-i\mu u}u(q-u)U(q,u)U(u,0)xdu\\ &= e^{-i\mu kq}(\int_0^q e^{-i\mu u}u(q-u)du)V^{n-k+1}x\\ &= M(\mu,q)e^{-i\mu(n+1)q}e^{i\mu(n-k+1)q}V^{n-k+1}x,\end{aligned}$$

where

$$M(\mu,q)=\int_0^q u(q-u)e^{-i\mu u}du\neq 0.$$

We return in (6) and obtain

$$\sup_{n\in\mathbf{N}}\|\sum_{j=0}^{n+1}e^{i\mu jq}V^j\|<\infty,$$

that is, $r(V)<1$ and $\mathcal{U}$ is exponentially stable.

COROLLARY 5. *A q-periodic evolution family $\mathcal{U}$ on X is uniformly exponentially stable if and only if*

$$\sup_{t>0}\|\int_0^t U(t,\xi)f(\xi)d\xi\|<\infty,\quad\forall f\in AP(\mathbf{R}_+,X).$$

For the other proof of Corollary 5, see e.g. [2] and [14]. In the end we give a result about evolution families on the line. In this context,

$$\mathcal{U}=\{U(t,s):t\geq s\in\mathbf{R}\}$$

will be a q-periodic evolution family on $\mathbf{R}$. We shall use the same notations as in Section 2, with $\mathbf{R}_+$ replaced by $\mathbf{R}$ and variables such as s and t taking any value in $\mathbf{R}$. Let us consider the evolution semigroup $\mathbf{T}_{ap}$ associated to $\mathcal{U}$ on the space $AP(\mathbf{R},X)$. This semigroup is strongly continuous, see Naito and Minh, [10, Lemma 2].

COROLLARY 6. *Let $\mathcal{U} = \{U(t,s), t \geq s\}$ be a q- periodic evolution family of bounded linear operators on X and $\mathbf{T}_{ap}$ the evolution semigroup associated to $\mathcal{U}$ on the space $AP(\mathbf{R}, X)$. Then $\mathcal{U}$ is uniformly exponentially stable if and only if*

$$\sup_{t\geq 0} ||(\int_0^t e^{i\mu\xi} T_{ap}(\xi) f d\xi)(t)|| < \infty \quad \forall \mu \in \mathbf{R}, \quad \forall f \in P_q(\mathbf{R}_+, X).$$

Proof. For $t > 0$, we have

$$\begin{aligned}(\int_0^t e^{i\mu\xi} T_{ap}(\xi) f)(t) &= \int_0^t e^{i\mu\xi} U(t, t-\xi) f(t-\xi) d\xi \\ &= e^{i\mu t} \int_0^t e^{-i\mu\tau} U(t,\tau) f(\tau) d\tau.\end{aligned}$$

Now, from Theorem 4, it follows that the restriction $\mathcal{U}_0$ of $\mathcal{U}$ to the set $\{(t,s) : t \geq s \geq 0\}$ is uniformly exponentially stable. Let $N > 0$ and $\nu > 0$ such that

$$||U(t,s)|| \leq N e^{-\nu(t-s)}, \quad \forall t \geq s \geq 0.$$

Then for all real numbers u and v with $u \geq v$, we have

$$||U(u,v)|| = ||U(u+nq, v+nq)|| \leq N e^{-\nu(u-v)},$$

where $n \in \mathbf{N}$ is such that $v + nq \geq 0$, that is, $\mathcal{U}$ is uniformly exponentially stable.

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