NEW STEFFENSEN PAIRS

FENG QI AND JUN-XIANG CHENG

Abstract. In this article, using mathematical induction and analytic techniques, some new Steffensen pairs are established.

1. Introduction

Let f and g be integrable functions on $[a,b]$ such that f is decreasing and $0 \leqslant g(x) \leqslant 1$ for $x \in [a,b]$. Then

$$\int_{b-\lambda}^{b} f(x)\mathrm{d}x \leqslant \int_{a}^{b} f(x)g(x)\mathrm{d}x \leqslant \int_{a}^{a+\lambda} f(x)\mathrm{d}x, \tag{1}$$

where $\lambda = \int_a^b g(x)\mathrm{d}x$.

The inequality (1) is called Steffensen's inequality. For more information, please see [3, 4, 14].

In [1], its discrete analogue of inequality (1) was proved: Let $\{x_i\}_{i=1}^n$ be a decreasing finite sequence of nonnegative real numbers, $\{y_i\}_{i=1}^n$ be a finite sequence of real numbers such that $0 \leqslant y_i \leqslant 1$ for $1 \leqslant i \leqslant n$. Let $k_1, k_2 \in \{1, 2, \dots, n\}$ be such that $k_2 \leqslant \sum_{i=1}^{n} y_i \leqslant k_1$. Then

$$\sum_{i=n-k_2+1}^{n} x_i \leqslant \sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{k_1} x_i. \tag{2}$$

As a direct consequence of inequality (2), we have: Let $\{x_i\}_{i=1}^n$ be nonnegative real numbers such that $\sum_{i=1}^{n} x_i \leqslant A$ and $\sum_{i=1}^{n} x_i^2 \geqslant B^2$, where A and B are positive real numbers. Let $k \in \{1, 2, \dots, n\}$ be such that $k \geqslant \frac{A}{B}$. Then there are k numbers among $x_1, x_2, \dots, x_n$ whose sum is bigger than or equals to B.

The so-called Steffensen pair was defined in [2] as follows:

Date: July 13, 2000.

1991 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Steffensen inequality, Steffensen pair, generalized weighted mean values, extended mean values, mathematical induction, absolutely monotonic function.

The authors were supported in part by NSF of Henan Province (no. 004051800), SF for Pure Research of the Education Committee of Henan Province (no. 1999110004), and Doctor Fund of Jiaozuo Institute of Technology, The People's Republic of China.

Definition 1. Let $\varphi : [c,\infty) \to [0,\infty)$ and $\tau : (0,\infty) \to (0,\infty)$ be two strictly increasing functions, $c \geqslant 0$, let $\{x_i\}_{i=1}^n$ be a finite sequence of real numbers such that $x_i \geqslant c$ for $1 \leqslant i \leqslant n$, A and B be positive real numbers, and $\sum_{i=1}^n x_i \leqslant A$, $\sum_{i=1}^n \varphi(x_i) \geqslant \varphi(B)$. If, for any $k \in \{1,2,\dots,n\}$ such that $k \geqslant \tau(\frac{A}{B})$, there are k numbers among $x_1,\dots,x_n$ whose sum is not less than B, then we call (φ,τ) a Steffensen pair on $[c,\infty)$.

In [1] and [2], the following Steffensen pairs were found respectively:

$$(x^\alpha, x^{1/(\alpha-1)}), \quad \alpha \geqslant 2, \quad x \in [0,\infty);$$
$$(x\exp(x^\alpha - 1), (1+\ln x)^{1/\alpha}), \quad \alpha \geqslant 1, \quad x \in [1,\infty).$$

Let a and b be real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geqslant e$. Define

$$\varphi(x) = \begin{cases} \dfrac{x^{1+\ln b} - x^{1+\ln a}}{\ln x}, & x > 1; \\ \ln b - \ln a, & x = 1, \end{cases}$$
$$\tau(x) = x^{1/\ln\sqrt{ab}}.$$

Then it was verified in [2] that (φ,τ) is a Steffensen pair on $[1,\infty)$.

In this article, we will establish some new Steffensen pairs, that is

Theorem 1. *If a and b are real numbers satisfying $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} \geqslant e$, then*

$$\left(x\int_a^b t^{\ln x - 1}\,\mathrm{d}t, x^{1/\ln\sqrt{ab}} \right) \tag{3}$$

is a Steffensen pair on $[1,+\infty)$.

If a and b are real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geqslant e$, then

$$\left(x\int_a^b (\ln t)^n t^{\ln x - 1}\,\mathrm{d}t, x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1} - (\ln a)^{n+1}}{(\ln b)^{n+2} - (\ln a)^{n+2}}} \right) \tag{4}$$

are Steffensen pairs on $[1,+\infty)$ for any positive integer n.

Remark 1. This theorem generalizes the Proposition 2 in [2].

2. Lemmas

Lemma 1 ([2]). *Let $\psi : [c,\infty) \to [0,\infty)$ be increasing and convex, $c \geqslant 0$. Assume that ψ satisfies $\psi(xy) \geqslant \psi(x)g(y)$ for all $x \geqslant c$ and $y \geqslant 1$, where $g : [1,\infty) \to [0,\infty)$ is strictly increasing. Set $\varphi(x) = x\psi(x)$, $\tau(x) = g^{-1}(x)$, where g^{-1} is the inverse function of g. Then (φ,τ) is a Steffensen pair on $[c,\infty)$.*

Let $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} > e$. Define

$$h(x) = \begin{cases} \dfrac{b^x - a^x}{x}, & x \neq 0; \\ \ln b - \ln a, & x = 0. \end{cases} \tag{5}$$

It can be represented in integral form in [5]—[13] as follows

$$h(x) = \int_a^b t^{x-1} \mathrm{d}t, \quad x \in \mathbb{R}. \tag{6}$$

It had been verified in [10] that the function $h(x)$ is absolutely and regularly monotonic on $(-\infty, +\infty)$ for $b > a > 1$, or on $(0, +\infty)$ for $b > a^{-1} > 1$, completely and regularly monotonic on $(-\infty, +\infty)$ for $0 < a < b < 1$, or on $(-\infty, 0)$ for $1 < b < a^{-1}$. Furthermore, $h(x)$ is absolutely convex on $(-\infty, +\infty)$.

A function $f(t)$ is said to be absolutely monotonic on (c, d) if it has derivatives of all orders and $f^{(k)}(t) \geqslant 0$ for $t \in (c, d)$ and $k \in \mathbb{N}$. For information of absolutely (completely, regularly, respectively) monotonic (convex, respectively) function, please refer to [4, 6, 10].

Lemma 2. *For $x \geqslant 0$ and $n \geqslant 0$, we have*

$$h^{(n+1)}(x) \geqslant h^{(n)}(x). \tag{7}$$

Proof. It is clear that

$$h^{(n)}(x) = \int_a^b t^{x-1} (\ln t)^n \mathrm{d}t. \tag{8}$$

By the Tchebysheff's integral inequality or by Cauchy-Schwarz-Buniakowski inequality as in [5]—[13], we have

$$[h^{(n+1)}(x)]^2 \leqslant h^{(n)}(x) h^{(n+2)}(x). \tag{9}$$

Since the extended mean values $E(r, s; u, v)$ defined in [5, 9, 12] by

$$E(r, s; u, v) = \left[\frac{r}{s} \cdot \frac{u^s - v^s}{u^r - v^r}\right]^{1/(s-r)}, \qquad rs(r-s)(u-v) \neq 0; \tag{10}$$

$$E(r, 0; u, v) = \left[\frac{u^r - v^r}{\ln u - \ln v} \cdot \frac{1}{r}\right]^{1/r}, \qquad r(v-u) \neq 0; \tag{11}$$

$$E(r, r; u, v) = \mathrm{e}^{-1/r} \left(\frac{u^{u^r}}{v^{v^r}}\right)^{1/(u^r - v^r)}, \qquad r(u-v) \neq 0; \tag{12}$$

$$E(0, 0; u, v) = \sqrt{uv}, \qquad u \neq v;$$

$$E(r, s; u, u) = u, \qquad u = v;$$

are increasing with r and s for fixed positive numbers u and v, then, for every $y \geqslant 0$, the function $F(x) = \frac{h(x+y)}{h(x)}$ is increasing with x. Therefore

$$F'(x) = \frac{h'(x+y)h(x) - h(x+y)h'(x)}{[h(x)]^2} \geqslant 0,$$

hence

$$h'(x+y)h(x) - h(x+y)h'(x) \geqslant 0 \tag{13}$$

holds for all x and $y \geqslant 0$.

Taking $x = 0$ in (13), for all $y \geqslant 0$, we obtain

$$h'(y)h(0) - h(y)h'(0) \geqslant 0. \tag{14}$$

Since $h'(0) = h(0)\ln\sqrt{ab}$ and $\sqrt{ab} \geqslant e$, we have $h'(0) \geqslant h(0)$, and $h'(y) \geqslant h(y)$ for $y \geqslant 0$.

Note that inequality (13) can also be obtained from Lemma 4 in [12]: The functions $\frac{h^{(2(k+i)+1)}(t)}{h^{(2k)}(t)}$ are increasing with respect to t for i and k being nonnegative integers.

By mathematical induction, assume that $h^{(n+1)}(x) \geqslant h^{(n)}(x)$ for $n > 1$ and $x \geqslant 0$. Then, from inequality (9), we obtain

$$h^{(n)}(x)h^{(n+1)}(x) \leqslant [h^{(n+1)}(x)]^2 \leqslant h^{(n)}(x)h^{(n+2)}(x), \tag{15}$$

therefore

$$h^{(n+1)}(x) \leqslant h^{(n+2)}(x).$$

The proof is completed. □

3. Proof of Theorem 1

Now we give a proof of Theorem 1.

Set $\psi(x) = h^{(n)}(\ln x)$ for $x \geqslant 1$ and $n \geqslant 0$. Direct computation yields that $\psi'(x) = \frac{h^{(n+1)}(\ln x)}{x} > 0$ and $\psi''(x) = \frac{h^{(n+2)}(\ln x) - h^{(n+1)}(\ln x)}{x^2} \geqslant 0$. Hence $\psi(x)$ is increasing and convex.

Let $u, v, r, s \in \mathbb{R}$, let $p \not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval between x and y. Then the generalized weighted mean values $M_{p,f}(r, s; u, v)$ of the function f with weight p and two parameters r and s are defined in [6] by

$$M_{p,f}(r, s; u, v) = \left(\frac{\int_u^v p(t)f^s(t)\mathrm{d}t}{\int_u^v p(t)f^r(t)\mathrm{d}t}\right)^{1/(s-r)}, \qquad (r-s)(u-v) \neq 0; \tag{16}$$

$$M_{p,f}(r, r; u, v) = \exp\left(\frac{\int_u^v p(t)f^r(t)\ln f(t)\mathrm{d}t}{\int_u^v p(t)f^r(t)\mathrm{d}t}\right), \qquad u - v \neq 0; \tag{17}$$

$$M(r, s; u, u) = f(u).$$

¿From the Cauchy-Schwarz-Buniakowski inequality and standard argument, it was obtained in [13] that: The generalized weighted mean values $M_{p,f}(r,s;u,v)$ are increasing with both r and s for any given continuous nonnegative weight p, continuous positive function f, and fixed real numbers u and v. Then, if $b>a>1$, for $x,y\geqslant 0$ and $n\geqslant 1$, we have

$$\frac{h^{(n)}(x+y)}{h^{(n)}(x)}=\frac{\int_a^b t^{x+y-1}(\ln t)^n \mathrm{d}t}{\int_a^b t^{x-1}(\ln t)^n \mathrm{d}t}\geqslant \exp\left(y\cdot\frac{n+1}{n+2}\cdot\frac{(\ln b)^{n+2}-(\ln a)^{n+2}}{(\ln b)^{n+1}-(\ln a)^{n+1}}\right). \tag{18}$$

Therefore, for $x,y\geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)}=\frac{h^{(n)}(\ln(xy))}{h^{(n)}(\ln x)}=\frac{h^{(n)}(\ln x+\ln y)}{h^{(n)}(\ln x)}\geqslant y^{\frac{n+1}{n+2}\cdot\frac{(\ln b)^{n+2}-(\ln a)^{n+2}}{(\ln b)^{n+1}-(\ln a)^{n+1}}}. \tag{19}$$

Let $g(x)=x^{\frac{n+1}{n+2}\cdot\frac{(\ln b)^{n+2}-(\ln a)^{n+2}}{(\ln b)^{n+1}-(\ln a)^{n+1}}}$ for $x\geqslant 1$, then $g^{-1}(x)=x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1}-(\ln a)^{n+1}}{(\ln b)^{n+2}-(\ln a)^{n+2}}}$, $x\in[1,+\infty)$. By Lemma 1, (φ,τ), where $\varphi(x)=x\psi(x)=xh^{(n)}(\ln x)=x\int_a^b(\ln t)^n t^{\ln x-1}\mathrm{d}t$ and $\tau(x)=x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1}-(\ln a)^{n+1}}{(\ln b)^{n+2}-(\ln a)^{n+2}}}$ for $x\geqslant 1$ and $n\geqslant 0$, are Steffensen pairs on $[1,+\infty)$ for any given $n\geqslant 0$.

If a and b are real numbers satisfying $b>a>1$ or $b>a^{-1}>1$, and $\sqrt{ab}\geqslant e$, then, for $x,y\geqslant 0$, we have

$$\frac{h(x+y)}{h(x)}\geqslant\left(\sqrt{ab}\right)^y. \tag{20}$$

Therefore, for $x,y\geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)}=\frac{h(\ln(xy))}{h(\ln x)}=\frac{h(\ln x+\ln y)}{h(\ln x)}\geqslant y^{\ln\sqrt{ab}}. \tag{21}$$

Let $g(x)=x^{\ln\sqrt{ab}}$ for $x\geqslant 1$, then $g^{-1}(x)=x^{1/\ln\sqrt{ab}}$, $x\in[1,+\infty)$. By Lemma 1, (φ,τ), where $\varphi(x)=x\psi(x)=xh(\ln x)=x\int_a^b t^{\ln x-1}\mathrm{d}t$ and $\tau(x)=x^{1/\ln\sqrt{ab}}$ for $x\geqslant 1$, is a Steffensen pair on $[1,+\infty)$.

The proof is complete.

Remark 2. If considering the function $\int_x^y p(u)f^t(u)\mathrm{d}u$, then more new Steffensen pairs can be obtained. We will discuss this in a subsequent paper [8].

References

[1] J.-C. Evard and H. Gauchman, *Steffensen type inequalities over general measure spaces*, Analysis **17** (1997), 301-322.

[2] H. Gauchman, *Steffensen pairs and associated inequalities*, Journal of Inequalities and Applications **5** (2000), no. 1, 53–61

[3] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[4] D. S. Mitrinović, J. E. Pečaric and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[5] Josip Pečarić, Feng Qi, V. Šimić, and Sen-Lin Xu, *Refinements and extensions of an inequality*, III, Journal of Mathematical Analysis and Applications **227** (1998), no. 2, 439–448.

[6] Feng Qi, *Generalized weighted mean values with two parameters*, Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences **454** (1998), no. 1978, 2723–2732.

[7] Feng Qi, *On a two-parameter family of nonhomogeneous mean values*, Tamkang Journal of Mathematics **29** (1998), no. 2, 155–163.

[8] Feng Qi and Jun-Xiang Cheng, *On Steffensen pairs*, RGMIA Research Report Collection **3** (2000). http://rgmia.vu.edu.au.

[9] Feng Qi and Qiu-Ming Luo, *A simple proof of monotonicity for extended mean values*, Journal of Mathematical Analysis and Applications **224** (1998), 356–359.

[10] Feng Qi and Sen-Lin Xu, *The function* $(b^x - a^x)/x$*: Inequalities and properties*, Proceedings of the American Mathematical Society **126** (1998), 3355–3359.

[11] Feng Qi and Sen-Lin Xu, *Refinements and extensions of an inequality*, II, Journal of Mathematical Analysis and Applications **211** (1997), 616–620.

[12] Feng Qi, Sen-Lin Xu, and Lokenath Debnath, *A new proof of monotonicity for extended mean values*, International Journal of Mathematics and Mathematical Sciences **22** (1999), no. 2, 415–420.

[13] Feng Qi and Shi-Qin Zhang, *Note on monotonicity of generalized weighted mean values*, Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences **455** (1999), no. 1989, 3259–3260.

[14] J. F. Steffensen, *On certain inequalities and methods of approximation*, J. Inst. Actuaries **51** (1919), 274-297.

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China

E-mail address: qifeng@jzit.edu.cn

URL: http://rgmia.vu.edu.au/qi.html or http://rgmia.vu.edu.au/Members/FQi.htm

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China