ON STEFFENSEN PAIRS

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Abstract. In this article, by mathematical induction and properties of the generalized weighted mean values, some general Steffensen pairs are established.

1. Introduction

Let f and g be integrable functions on $[a,b]$ such that f is decreasing and $0 \leqslant g(x) \leqslant 1$ for $x \in [a,b]$. Then

$$\int_{b-\lambda}^{b} f(x)\mathrm{d}x \leqslant \int_{a}^{b} f(x)g(x)\mathrm{d}x \leqslant \int_{a}^{a+\lambda} f(x)\mathrm{d}x, \tag{1}$$

where $\lambda = \int_a^b g(x)\mathrm{d}x$.

The inequality (1) is called Steffensen's inequality in [3, 8].

In [1], its discrete analogue of the inequality (1) was proved: Let $\{x_i\}_{i=1}^n$ be a decreasing finite sequence of nonnegative real numbers, $\{y_i\}_{i=1}^n$ be a finite sequence of real numbers such that $0 \leqslant y_i \leqslant 1$ for $1 \leqslant i \leqslant n$. Let $k_1, k_2 \in \{1, 2, \dots, n\}$ be such that $k_2 \leqslant \sum_{i=1}^{n} y_i \leqslant k_1$. Then

$$\sum_{i=n-k_2+1}^{n} x_i \leqslant \sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{k_1} x_i. \tag{2}$$

As a direct consequence of inequality (2), we obtain: Let $\{x_i\}_{i=1}^n$ be a finite sequence of non-negative real numbers such that $\sum_{i=1}^{n} x_i \leqslant A$ and $\sum_{i=1}^{n} x_i^2 \geqslant B^2$, where A and B are positive real numbers. Let $k \in \{1, 2, \dots, n\}$ be such that $k \geqslant \frac{A}{B}$. Then there are k numbers among $\{x_i\}_{i=1}^n$ whose sum is not less than B.

From above results, the so-called Steffensen pair was defined in [2] by Dr. H. Gauchman as follows:

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Definition 1. Let $\varphi : [c, \infty) \to [0, \infty)$ and $\tau : (0, \infty) \to (0, \infty)$ be two strictly increasing functions, $c \geqslant 0$, let $\{x_i\}_{i=1}^n$ be a finite sequence of real numbers such that $x_i \geqslant c$ for all i, A and B be positive real numbers, $\sum_{i=1}^n x_i \leqslant A$, and $\sum_{i=1}^n \varphi(x_i) \geqslant \varphi(B)$. If, for any $k \in \{i\}_{i=1}^n$ satisfying $k \geqslant \tau(\frac{A}{B})$, there are k numbers among $\{x_i\}_{i=1}^n$ whose sum is not less than B, then we call (φ, τ) a Steffensen pair on $[c, \infty)$.

In [1] and [2], the following Steffensen pairs were found:

$$(x^\alpha, x^{1/(\alpha-1)}), \quad \alpha \geqslant 2, \quad x \in [0, \infty);$$

$$(x \exp(x^\alpha - 1), (1 + \ln x)^{1/\alpha}), \quad \alpha \geqslant 1, \quad x \in [1, \infty).$$

It was verified in [5] that: Let a and b be real numbers satisfying $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} \geqslant e$, then

$$\left(x \int_a^b t^{\ln x - 1} \mathrm{d}t, x^{1/\ln\sqrt{ab}}\right) \tag{3}$$

is a Steffensen pair on $[1, +\infty)$. If a and b are real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geqslant e$, then

$$\left(x \int_a^b (\ln t)^n t^{\ln x - 1} \mathrm{d}t, x^{\frac{n+2}{n+1} \cdot \frac{(\ln b)^{n+1} - (\ln a)^{n+1}}{(\ln b)^{n+2} - (\ln a)^{n+2}}}\right) \tag{4}$$

are Steffensen pairs on $[1, +\infty)$ for any positive integer n.

In this article, we will establish more general Steffensen pairs, that is

Theorem 1. *Let $a, b \in \mathbb{R}$, let $p \not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval $[a, b]$.*

(i) *If inequality*

$$\int_a^b p(u) \mathrm{d}u \leqslant \int_a^b p(u) \ln f(u) \mathrm{d}u \tag{5}$$

holds, then

$$\left(x \int_a^b p(u)[f(u)]^{\ln x} du, x^{\frac{\int_a^b p(u) du}{\int_a^b p(u) \ln f(u) du}}\right) \tag{6}$$

is a Steffensen pair on $[1, +\infty)$.

(ii) *If $f(u) \geqslant 1$ and inequality (5) holds, then*

$$\left(x \int_a^b p(u)[f(u)]^{\ln x} [\ln f(u)]^n du, x^{\frac{\int_a^b p(u)[\ln f(u)]^n du}{\int_a^b p(u)[\ln f(u)]^{n+1} du}}\right) \tag{7}$$

are Steffensen pairs on $[1, +\infty)$ for any positive integer n.

Remark 1. This theorem generalizes the Proposition 2 in [2] and the related results in [5].

2. Lemmas

Lemma 1 ([2]). *Let $\psi:[c,\infty)\to[0,\infty)$ be increasing and convex, $c\geqslant 0$. Assume that ψ satisfies $\psi(xy)\geqslant\psi(x)g(y)$ for all $x\geqslant c$ and $y\geqslant 1$, where $g:[1,\infty)\to[0,\infty)$ is strictly increasing. Set $\varphi(x)=x\psi(x)$, $\tau(x)=g^{-1}(x)$, where g^{-1} is the inverse function of g. Then (φ,τ) is a Steffensen pair on $[c,\infty)$.*

Define

$$h(t)=\int_a^b p(u)f^t(u)du,\quad t\in\mathbb{R},\tag{8}$$

where $p(u)$ is a nonnegative and continuous function, $f(u)$ a positive and continuous function on the interval $[a,b]$, and $a,b\in\mathbb{R}$.

It is clear [4, 7] that, if $f(u)\geqslant 1$ on $[a,b]$, then

$$h^{(n)}(t)=\int_a^b p(u)f^t(u)[\ln f(u)]^n du\geqslant 0,\tag{9}$$

that is, $h(t)$ is an absolutely monotonic function, see [3, 4].

By the Cauchy-Schwarz-Buniakowski inequality, it is easy to obtain

Lemma 2. *For $n\geqslant 0$, if $f(u)\geqslant 1$ on $[a,b]$, then we have*

$$[h^{(n+1)}(x)]^2\leqslant h^{(n)}(x)h^{(n+2)}(x),\quad x\in\mathbb{R}.\tag{10}$$

Let $a,b,r,s\in\mathbb{R}$, let $p\not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval between a and b. Then the generalized weighted mean values $M_{p,f}(r,s;a,b)$ of the function f with weight p and two parameters r and s are defined in [4] by

$$M_{p,f}(r,s;a,b)=\left(\frac{\int_a^b p(u)f^s(u)\mathrm{d}u}{\int_a^b p(u)f^r(u)\mathrm{d}u}\right)^{1/(s-r)}=\left(\frac{h(s)}{h(r)}\right)^{1/(s-r)},\qquad (r-s)(a-b)\neq 0;\tag{11}$$

$$M_{p,f}(r,r;a,b)=\exp\left(\frac{\int_a^b p(u)f^r(u)\ln f(u)\mathrm{d}u}{\int_a^b p(u)f^r(u)\mathrm{d}u}\right)=\exp\left(\frac{h'(r)}{h(r)}\right),\qquad a-b\neq 0;\tag{12}$$

$$M(r,s;a,a)=f(a).$$

From the Cauchy-Schwarz-Buniakowski inequality again and standard argument, we have

Lemma 3 ([7]). *The generalized weighted mean values $M_{p,f}(r,s;a,b)$ are increasing with both r and s for any given continuous nonnegative weight p and continuous positive function f.*

Lemma 4. *For $n\geqslant 0$ and $x\geqslant 0$, if*

$$\int_a^b p(u)\mathrm{d}u\leqslant\int_a^b p(u)\ln f(u)\mathrm{d}u,\tag{13}$$

then we have

$$h^{(n+1)}(x) \geqslant h^{(n)}(x). \tag{14}$$

Proof. By Lemma 3, the mean values

$$\left(\frac{h(x+y)}{h(x)}\right)^{1/y}$$

are increasing with respect to x and y, then the function

$$F(x) = \frac{h(x+y)}{h(x)}$$

is increasing with x for fixed $y \geqslant 0$. Therefore

$$F'(x) = \frac{h'(x+y)h(x) - h(x+y)h'(x)}{[h(x)]^2} \geqslant 0.$$

Hence, the inequality

$$h'(x+y)h(x) - h(x+y)h'(x) \geqslant 0 \tag{15}$$

holds for all x and all $y \geqslant 0$.

Note that the inequality (15) can also be obtained from the Lemma in [6].

Taking $x = 0$ in inequality (15), we obtain

$$h'(y)h(0) - h(y)h'(0) \geqslant 0 \tag{16}$$

for all $y \geqslant 0$, and

$$h(0) = \int_a^b p(u)\mathrm{d}u, \tag{17}$$

$$h'(0) = \int_a^b p(u)\ln f(u)\mathrm{d}u. \tag{18}$$

Since inequality (13) means that $h'(0) \geqslant h(0)$, thus $h'(y) \geqslant h(y)$ for all $y \geqslant 0$.

By mathematical induction, assume that $h^{(n+1)}(x) \geqslant h^{(n)}(x)$ for $n \geqslant 2$ and $x \geqslant 0$. Then, from Lemma 2, we obtain

$$h^{(n)}(x)h^{(n+1)}(x) \leqslant [h^{(n+1)}(x)]^2 \leqslant h^{(n)}(x)h^{(n+2)}(x), \tag{19}$$

therefore

$$h^{(n+1)}(x) \leqslant h^{(n+2)}(x).$$

The proof is completed. □

3. New Steffensen Pairs

Now we give a proof of Theorem 1.

Set $\psi(x) = h^{(n)}(\ln x)$ for $x \geqslant 1$ and $n \geqslant 0$. Direct computation yields that $\psi'(x) = \frac{h^{(n+1)}(\ln x)}{x} > 0$ and $\psi''(x) = \frac{h^{(n+2)}(\ln x) - h^{(n+1)}(\ln x)}{x^2} \geqslant 0$. Hence $\psi(x)$ is increasing and convex.

Since $f(u) \geqslant 1$, for $n \geqslant 1$, by Lemma 3, we have

$$\frac{h^{(n)}(x+y)}{h^{(n)}(x)} = \frac{\int_a^b p(u)[f(u)]^{x+y}[\ln f(u)]^n du}{\int_a^b p(u)[f(u)]^x[\ln f(u)]^n du} \geqslant \exp\left(y \cdot \frac{\int_a^b p(u)[\ln f(u)]^{n+1} du}{\int_a^b p(u)[\ln f(u)]^n du}\right). \tag{20}$$

Therefore, for $x, y \geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)} = \frac{h^{(n)}(\ln(xy))}{h^{(n)}(\ln x)} = \frac{h^{(n)}(\ln x + \ln y)}{h^{(n)}(\ln x)} \geqslant y^{\frac{\int_a^b p(u)[\ln f(u)]^{n+1} du}{\int_a^b p(u)[\ln f(u)]^n du}}. \tag{21}$$

Let $g(x) = x^{\frac{\int_a^b p(u)[\ln f(u)]^{n+1} du}{\int_a^b p(u)[\ln f(u)]^n du}}$ for $x \geqslant 1$, then $g^{-1}(x) = x^{\frac{\int_a^b p(u)[\ln f(u)]^{n} du}{\int_a^b p(u)[\ln f(u)]^{n+1} du}}$, $x \in [1, +\infty)$.

By Lemma 1, (φ, τ), where $\varphi(x) = x\psi(x) = xh^{(n)}(\ln x) = x\int_a^b p(u)[f(u)]^{\ln x}[\ln f(u)]^n du$ and $\tau(x) = x^{\frac{\int_a^b p(u)[\ln f(u)]^{n} du}{\int_a^b p(u)[\ln f(u)]^{n+1} du}}$ for $x \geqslant 1$ and $n \geqslant 1$, are Steffensen pairs on $[1, +\infty)$ for any given $n \geqslant 1$.

For $n = 0$, by Lemma 3, we have

$$\frac{h(x+y)}{h(x)} = \frac{\int_a^b p(u)[f(u)]^{x+y} du}{\int_a^b p(u)[f(u)]^x du} \geqslant \exp\left(y \cdot \frac{\int_a^b p(u)\ln f(u) du}{\int_a^b p(u) du}\right). \tag{22}$$

Therefore, for $x, y \geqslant 1$,

$$\frac{\psi(xy)}{\psi(x)} = \frac{h(\ln(xy))}{h(\ln x)} = \frac{h(\ln x + \ln y)}{h(\ln x)} \geqslant y^{\frac{\int_a^b p(u)\ln f(u) du}{\int_a^b p(u) du}}. \tag{23}$$

Let $g(x) = x^{\frac{\int_a^b p(u)\ln f(u) du}{\int_a^b p(u) du}}$ for $x \geqslant 1$, then $g^{-1}(x) = x^{\frac{\int_a^b p(u) du}{\int_a^b p(u)\ln f(u) du}}$, $x \in [1, +\infty)$.

By Lemma 1, (φ, τ), where $\varphi(x) = x\psi(x) = xh(\ln x) = x\int_a^b p(u)[f(u)]^{\ln x} du$ and $\tau(x) = x^{\frac{\int_a^b p(u) du}{\int_a^b p(u)\ln f(u) du}}$ for $x \geqslant 1$ and $n \geqslant 1$, are Steffensen pairs on $[1, +\infty)$ for any given $n \geqslant 1$.

The proof is complete.

References

[1] J.-C. Evard and H. Gauchman, *Steffensen type inequalities over general measure spaces*, Analysis **17** (1997), 301-322.

[2] H. Gauchman, *Steffensen pairs and associated inequalities*, Journal of Inequalities and Applications **5** (2000), no. 1, 53–61

[3] D. S. Mitrinović, J. E. Pečaric and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[4] Feng Qi, *Generalized weighted mean values with two parameters*, Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences **454** (1998), no. 1978, 2723–2732.

[5] Feng Qi and Jun-Xiang Cheng, *New Steffensen pairs*, RGMIA Research Report Collection **3** (2000). http://rgmia.vu.edu.au.

[6] Feng Qi, Jia-Qiang Mei, Da-Feng Xia, and Sen-Lin Xu, *New proofs of weighted power mean inequalities and monotonicity for generalized weighted mean values*, Mathematical Inequalities and Applications **3** (2000), no. 3, in the press.

[7] Feng Qi and Shi-Qin Zhang, *Note on monotonicity of generalized weighted mean values*, Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences **455** (1999), no. 1989, 3259–3260.

[8] J. F. Steffensen, *On certain inequalities and methods of approximation*, J. Inst. Actuaries **51** (1919), 274-297.

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