NEW GENERALIZATIONS OF THE TELYAKOVSKII'S INEQUALITIES

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Abstract. Generalization of the theorems of Telyakovski [1] have been obtained, by considering the condition $S_{p\alpha r},\ 1 < p \leq 2,\ \alpha \geq 0,\ r \in \{0, 1, 2, \ldots [\alpha]\}$ instead of S.

1. Introduction and preliminaries

A sequence $\{a_n\}$ of real numbers satisfies condition S, or $\{a_n\} \in S$, if $a_n \to 0$ as $n \to \infty$ and there exists a sequence of numbers $\{A_n\}$ such that:

a) $A_n \downarrow 0$

b) $\sum_{n=1}^{\infty} A_n < \infty$.

c) $|\Delta a_n| \leq A_n$, for all n.

Now we define a stronger class $S_{p\alpha r}$ of numbers as follows:

A null sequence $\{a_n\}$ of real numbers belongs to the class $S_{p\alpha r}$ if for some $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots [\alpha]\}$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} n^{\alpha} A_n < \infty$, the condition $\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$, $1 < p \leq 2$ holds.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ and $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ be the well-known cosine and sine trigonometric series.

In 1973, Telyakovski [1] has proved the following Theorems:

Theorem A. *Let the coefficients of the series $f(x)$ satisfy the condition S. Then the series is a Fourier series and the following relation holds:*

$$\int_0^{\pi} |f(x)|\, dx \leq C \sum_{n=1}^{\infty} A_n\,,$$

where C is an absolute constant.

Theorem B. *Let the coefficients of the series $g(x)$ satisfy the condition S. Then the following relation holds for $p = 1, 2, 3, \ldots$.*

$$\int\limits_{\pi/(p+1)}^{\pi} |g(x)|\,dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n\right).$$

In particular $g(x)$ is a Fourier series iff $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

For the proof of our new theorems we need the following lemma:

Lemma. [2]. *Let r be a nonnegative integer, and $x \in (0, \pi]$, where $n \geq 1$. Then*

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{\left(n+\dfrac{1}{2}\right)^k \sin\left[\left(n+\dfrac{1}{2}\right)x + \dfrac{k\pi}{2}\right]}{\left[\sin\left(\dfrac{x}{2}\right)\right]^{r+1-k}} \varphi_k(x) + \frac{\left(n+\dfrac{1}{2}\right)^r \sin\left[\left(n+\dfrac{1}{2}\right)x + \dfrac{r\pi}{2}\right]}{2\sin\left(\dfrac{x}{2}\right)},$$

where the same φ_k denotes various analytical function of x, independent of n, and $D_n(x)$ is the Dirichlet kernel.

2. Main results

We prove the following theorems:

Theorem 1. *Let the coefficients of the series $f(x)$ satisfy the conditions $S_{p\alpha r}$, $1 < p \leq 2$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots [\alpha]\}$. Then the series $f^{(r)}(x)$ is a Fourier series and the following relation hold:*

$$\int\limits_{0}^{\pi} |f^{(r)}(x)|\,dx \leq M_p \sum_{n=0}^{\infty} n^{\alpha} A_n\,,$$

where M_p, is an absolute constant depends only on p.

Proof. We have:

$$\sum_{k=1}^{n} |\Delta(k^r a_k)| = \sum_{k=1}^{n} |[(k+1)^r a_{k+1} - k^r a_{k+1}] + [k^r a_{k+1} - k^r a_k]| =$$

$$= \sum_{k=1}^{n} |[\Delta(k^r) a_{k+1} + k^r \Delta a_k| \leq \sum_{k=1}^{n} k^{r-1} |a_{k+1}| + \sum_{k=1}^{n} k^r |\Delta a_k|\,.$$

Applying Abel's transformation, we have:

$$\sum_{k=1}^{n} k^{r-1} |a_{k+1}| = \sum_{k=1}^{n-1} \Delta |a_{k+1}| \sum_{j=1}^{k} j^{r-1} + |a_{n+1}| \sum_{j=1}^{n} j^{r-1} \leq$$

$$\leq \sum_{k=1}^{n-1} |\Delta a_{k+1}|\, k^r + |a_{n+1}|\, n^r =$$

$$= \sum_{k=1}^{n-1} |\Delta a_{k+1}|\, k^r + n^r |\Delta a_{n+1} + \Delta a_{n+2} + \cdots| \leq$$

$$\leq \sum_{k=1}^{n-1} |\Delta a_{k+1}|\, k^r + (n+1)^r \sum_{k=n+1}^{\infty} |\Delta a_k| \leq$$

$$\leq \sum_{k=1}^{n-1} |\Delta a_{k+1}|\, k^r + \sum_{k=n+1}^{\infty} k^r |\Delta a_k|\,,$$

and

$$\sum_{k=1}^{n} k^r |\Delta a_k| = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} j^r + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} j^r \leq$$

$$\leq \sum_{k=1}^{n-1} (\Delta A_k) k^{1+\alpha} \left[k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right]^{1/p} +$$

$$+ n^{1+\alpha} A_n \left[n^{p(r-\alpha)-1} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right]^{1/p} =$$

$$= O(1) \left[\sum_{k=1}^{n-1} (\Delta A_k) k^{1+\alpha} + n^{1+\alpha} A_n \right] =$$

$$= O(1) \left\{ \sum_{k=1}^{n} [k^{\alpha+1} - (k-1)^{\alpha+1}] A_k - n^{1+\alpha} A_n + n^{1+\alpha} A_n \right\} =$$

$$= O(1) \sum_{k=1}^{n} k^{\alpha} A_k\,.$$

Thus

$$\sum_{k=1}^{\infty} |\Delta (k^r a_k)| \leq O(1) \sum_{k=1}^{\infty} k^{\alpha} A_k < \infty \quad \text{i.e} \quad \lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x)\,.$$

For $r = 0$, we have: $f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x)$ $(*)$.

From inequality $|D_n^{(r)}(x)| \leq C\,\frac{n^r}{x}$, we have that $\sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0, \pi)$.

Thus representation $(*)$ implies

$$f^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)\,.$$

Now applications of Abel's transformation yield,

$$\int_0^{\pi} |f^{(r)}(x)|\, dx = \int_0^{\pi} \left| \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq$$

$$\leq \sum_{k=0}^{\infty}(\Delta A_k)\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}D_j^{(r)}(x)\right|\,dx+A_n\int_0^{\pi}\left|\sum_{j=0}^{n}\frac{\Delta a_j}{A_j}D_j^{(r)}(x)\right|\,dx\,.$$

Then,

$$\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}D_j^{(r)}(x)\right|\,dx=\int_0^{\pi/k}+\int_{\pi/k}^{\pi}=I_k+J_k\,.$$

Applying the inequality $D_n^{(r)}(x)=O(n^{r+1})$, we have:

$$I_k\leq A\sum_{j=1}^{k}j^r\frac{|\Delta a_j|}{A_j}\leq Ak^{r+1}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p}=Ak^{1+\alpha}\left[k^{p(r-\alpha)-1}\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right]^{1/p},$$

where A is an absolute constant.

Let us estimate the second integral:

$$J_k=\int_{\pi/k}^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\right|\,dx\leq(\textit{Lemma})$$

$$\leq\int_{\pi/k}^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\left(\sum_{v=0}^{r-1}\frac{\left(j+\frac{1}{2}\right)^v\sin\left[\left(j+\frac{1}{2}\right)x+\frac{v\pi}{2}\right]}{\left[\sin\left(\frac{x}{2}\right)\right]^{r+1-v}}\varphi_k\right)\right|\,dx+$$

$$+\int_{\pi/k}^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\,\frac{\left(j+\frac{1}{2}\right)^r\sin\left[\left(j+\frac{1}{2}\right)x+\frac{r\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)}\right|\,dx=\lambda_k+\mu_k\,.$$

Since φ_k is bounded, it can be shown by Holder inequality that:

$$T_k=\int_{\pi/k}^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,\frac{(j+\frac{1}{2})^v\sin\left[\left(j+\frac{1}{2}\right)x+\frac{v\pi}{2}\right]}{\left[\sin\left(\frac{x}{2}\right)\right]^{r+1-v}}\varphi_k\right|\,dx\leq$$

$$\leq B\int_{\pi/k}^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,\frac{(j+\frac{1}{2})^v\sin\left[\left(j+\frac{1}{2}\right)x+\frac{v\pi}{2}\right]}{\left[\sin\left(\frac{x}{2}\right)\right]^{r+1-v}}\right|\,dx\leq$$

$$\leq B\left[\int_{\pi/k}^{\pi}\frac{dx}{\left[\sin\left(\frac{x}{2}\right)\right]^{(r+1-v)p}}\right]^{1/p}\left\{\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\left(j+\frac{1}{2}\right)^v\sin\left[\left(j+\frac{1}{2}\right)x+\frac{v\pi}{2}\right]\right|^q dx\right\}^{1/q},\ B>0,$$

where B is a positive constant dependent on r. Since

$$\int_{\pi/k}^{\pi} \frac{dx}{\left[\sin\left(\frac{x}{2}\right)\right]^{(r+1-v)p}} \leq \pi^{(r+1-v)p} \int_{\pi/k}^{\pi} \frac{dx}{x^{(r+1-v)p}} \leq \frac{\pi k^{(r+1-v)p-1}}{(r+1-v)p-1} \leq \frac{\pi k^{(r+1-v)p-1}}{p-1},$$

we have:

$$T_k \leq B\left(\frac{\pi}{p-1}\right)^{1/p} [k^{(r+1-v)p-1}]^{1/p} \times$$

$$\times \left\{\int_0^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j}\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]\right|^q dx\right\}^{1/q}.$$

Then using the Hausdorff-Young inequality we get:

$$\left\{\int_0^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j}\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]\right|^q dx\right\}^{1/q} \leq$$

$$\leq \left\{\int_0^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j}\left(j+\frac{1}{2}\right)^v e^{ijx}\right|^q dx\right\}^{1/q} = O\left[\left(\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} j^{vp}\right)^{1/p}\right].$$

Finally,

$$T_k \leq C_p [k^{(r+1-v)p-1}]^{1/p} \left(\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} j^{vp}\right)^{1/p} \leq C_p [k^{(r+1)p-1}]^{1/p} \left(\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p} =$$

$$= C_p k^{1+\alpha} \left[k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right]^{1/p},$$

where C_p is a positive constant which depends only on p. Since r is a finite value, we have:

$$\lambda_k = O_p\left\{k^{1+\alpha}\left[k^{p(r-\alpha)-1}\left(\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right)\right]^{1/p}\right\},$$

where O_p depends only on p. Similarly, we can get:

$$\mu_k = O_p\left\{k^{1+\alpha}\left[k^{p(r-\alpha)-1}\left(\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right)\right]^{1/p}\right\}.$$

Thus

$$\int_0^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx \leq A k^{1+\alpha}\left[k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\right]^{1/p} +$$

$$+O_p\left\{k^{1+\alpha}\left[k^{p(r-\alpha)-1}\left(\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right)\right]^{1/p}\right\}=O(k^{1+\alpha})+O_p(k^{1+\alpha})=O_p(k^{1+\alpha})\,.$$

Since $N^{1+\alpha}A_N=o(1)$, $N\to\infty$ we have:

$$\int\limits_0^\pi|f^{(r)}(x)|\,dx\le O_p(1)\left[\sum_{k=0}^{\infty}(\Delta A_k)k^{1+\alpha}+n^{1+\alpha}A_n\right]=$$

$$=O_p(1)\lim_{N\to\infty}\left[\sum_{k=0}^{N-1}(\Delta A_k)k^{1+\alpha}+n^{1+\alpha}A_n\right]=$$

$$=O_p(1)\lim_{N\to\infty}\left[\sum_{k=0}^{N}O(k^\alpha A_k)-N^{1+\alpha}A_N+n^{1+\alpha}A_n\right]=$$

$$=O_p(1)\left[\sum_{k=0}^{N}O(k^\alpha A_k)+n^{1+\alpha}A_n\right]\,.$$

On the other hand, by $n^{1+\alpha}A_n=o(1),\ n\to\infty$ the following inequality is satisfied:

$$\int\limits_0^\pi|f^{(r)}(x)|\,dx\le M_p\sum_{k=0}^{\infty}n^\alpha A_n\,,\quad M_p>0\,.$$

Theorem 2. *Let the coefficients of the series* $g(x)$ *satisfy the condition* $S_{p\alpha r}$, $1<p\le 2$, $\alpha\ge 0$, $r\in\{0,1,2,\dots[\alpha]\}$.

Then the following relation hold for $m=1,2.3,\dots$

$$\int\limits_{\pi/m+1}^{\pi}|g^{(r)}(x)|\,dx=\sum_{n=1}^{m}|a_n|\,n^{r-1}+O_p\left(\sum_{n=1}^{\infty}n^\alpha A_n\right)\,,$$

where O_p *depends only on* p.

In particular $g^{(r)}(x)$ *is a Fourier series iff* $\sum_{n=1}^{\infty}|a_n|\,n^{r-1}<\infty$.

Proof. We suppose that $a_0=0$ and $A_0=\max(|a_n|,A_1)$.

For $r=0$, applying the Abel's transformation, we have:

$$g(x)=\sum_{k=0}^{\infty}\Delta a_k\overline{D_k(x)}\qquad(**)$$

where

$$\overline{D_k(x)}=-\frac{\operatorname{ctg}\dfrac{x}{2}}{2}+\tilde{D}_k(x)=-\frac{\cos\left(k+\dfrac12\right)x}{2\sin\dfrac{x}{2}}\,,$$

and $\tilde{D}_n(x)$ is the conjugate Dirichlet kernel.

Since $-\cos\left(n+\frac{1}{2}\right)x = \sin\left[\left(n+\frac{1}{2}\right)x+\frac{3\pi}{2}\right]$, by the Lemma, we get:

$$\overline{D_n(x)}^{(r)} = \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x+\frac{k+3}{2}\pi\right]}{\left[\sin\left(\frac{x}{2}\right)\right]^{r+1-k}}\varphi_k(x)+\frac{\left(n+\frac{1}{2}\right)^{(r)}\sin\left[\left(n+\frac{1}{2}\right)x+\frac{r+3}{2}\pi\right]}{2\sin\left(\frac{x}{2}\right)},$$

where the same φ_k denotes various analytical function of x, independent of n.

Similarly as in the proof of the Theorem 1, we get that representation $(**)$ implies

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D_k(x)}^{(r)}.$$

Then,

$$\int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|\,dx = \int_{\pi/(m+1)}^{\pi} \left|\sum_{k=0}^{\infty} \Delta a_k\, \overline{D_k(x)}^{(r)}\right| dx =$$

$$= \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k \overline{D_k(x)}^{(r)}\right| dx + O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=j}^{\infty} \Delta a_k\, \overline{D_k(x)}^{(r)}\right| dx\right).$$

Let

$$I_1 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=j}^{\infty} \Delta a_k \overline{D_k(x)}^{(r)}\right| dx\,, \qquad I_2 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k \overline{D_k(x)}^{(r)}\right| dx\,.$$

Now applyng the inequality:

$$\left|\overline{D_k(x)}^{(r)}\right| = O\left(\frac{k^r}{x}\right)$$

we have:

$$I_2 = O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=j}^{j-1} \Delta a_k\right| j^r \frac{dx}{x}\right) = O\left[\sum_{j=1}^{m} |a_j|\, j^r \ln\left(1+\frac{1}{j}\right)\right] = O\left(\sum_{j=1}^{m} |a_j| j^{r-1}\right).$$

Application of Abel's transformation, yield,

$$\sum_{k=j}^{\infty} \Delta a_k \overline{D_k(x)}^{(r)} = \sum_{k=j}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{|\Delta a_i|}{A_i} \overline{D_k(x)}^{(r)} - A_j \sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \overline{D_i(x)}^{(r)}.$$

Let us estimate the following integral:

$$I_1 \le \sum_{j=1}^{m} \left[\sum_{k=j}^{\infty} (\Delta A_k) \int_{\pi/(j+1)}^{\pi} \left|\sum_{i=0}^{k} \frac{|\Delta a_i|}{A_i} \overline{D_i(x)}^{(r)}\right| dx + A_j \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{i=0}^{j} \frac{|\Delta a_i|}{A_i} \overline{D_i(x)}^{(r)}\right| dx\right].$$

Applying the Lemma and the same technique as in the proof of Theorem 1, we have:

$$J_k = \int_{\pi/(j+1)}^{\pi} \left|\sum_{i=0}^{k} \frac{|\Delta a_i|}{A_i} \overline{D_i(x)}^{(r)}\right| dx = O_p\left((k+1)^{\alpha}\right),$$

where O_p dependents only on p. But

$$I_j = \int\limits_{\pi/(j+1)}^{\pi/j} \left| \sum_{i=0}^{j} \frac{|\Delta a_i|}{A_i} \overline{D_i(x)}^{(r)} \right| dx = O\left[\int\limits_{\pi/(j+1)}^{\pi/j} j^r \left(\sum_{i=0}^{j} \frac{|\Delta a_i|}{A_i} \right) \frac{dx}{x} \right] =$$

$$= O\left[j^r \ \ln\left(1+\frac{1}{j}\right) \left(\sum_{i=0}^{j} \frac{|\Delta a_i|}{A_i} \right) \right] =$$

$$= O\left(j^{r-1} \sum_{i=0}^{j} \frac{|\Delta a_i|}{A_i} \right) = O\left[j^r \ \left(\frac{1}{j} \sum_{i=0}^{j} \frac{|\Delta a_i|^p}{A_i^p} \right)^{1/p} \right] =$$

$$= O\left[j^{\alpha} \left(j^{p(r-\alpha)-1} \sum_{i=0}^{j} \frac{|\Delta a_i|^p}{A_i^p} \right)^{1/p} \right] = O(j^{\alpha}) \, .$$

Now, we have:

$$I_1 \le \sum_{k=1}^{\infty} (\Delta A_k) J_k + M \sum_{j=1}^{\infty} j^{\alpha} A_j =$$

$$= O_p(1) \sum_{k=1}^{\infty} (\Delta A_k)(k+1)^{\alpha} + M \sum_{j=1}^{\infty} j^{\alpha} A_j = O_p \left(\sum_{j=1}^{\infty} j^{\alpha} A_j \right) \, .$$

Finally the inequality is satisfied.

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Новые обобщения неравенств Телияковского

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Впредлагаемой работе мы доказываем обопщение неравенств С. А. Теляковского [1]. Рассматривается более общее условие $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, 2, \dots [\alpha]\}$ вместо условия S.