IMPROVEMENT OF AN OSTROWSKI TYPE INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATION FOR SOME SPECIAL MEANS

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Abstract. We first improve two Ostrowski type inequalities for monotonic functions, then provide its application for special means.

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1. Introduction.

In [1], Dragomir established the following Ostrowski's inequality for monotonic mappings.

Theorem 1. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the following inequality*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| &\leq \frac{1}{b-a}\left\{[2x-(a+b)]f(x) + \int_a^b sgn(t-x)f(t)dt\right\} \\ &\leq \frac{1}{b-a}[(x-a)(f(x)-f(a)) + (b-x)(f(b)-f(x))] \\ &\leq \left[\frac{1}{2} + \frac{|x-((a+b)/2)|}{b-a}\right](f(b)-f(a)). \qquad (1.1)\end{aligned}$$

And the constant $1/2$ *is the best possible one.*

In [2], Dragomir, Pečarić and Wang generalized Theorem 1 and proved

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Theorem 2. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$ and t_1, t_2, $t_3 \in (a, b)$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
\le &(b - t_3)f(b) + (2t_2 - t_1 - t_3)f(t_2) - (t_1 - a)f(a) + \int_a^b T(x)f(x)dx \\
\le &(b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
&+(t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(a)) \\
\le &\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)), \qquad (1.2)
\end{aligned}$$

where $T(x) = sgn(t_1 - x)$, for $x \in [a, t_2]$, and $T(x) = sgn(t_3 - x)$, for $x \in [t_2, b]$.

In the present paper, we firstly improve the above results, and then provide its application for some special means.

2. Main Result.

We shall start with the following result.

Theorem 3. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$ and let t_1, t_2, $t_3 \in [a, b]$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
\le &\max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&(t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \qquad (2.1) \\
\le &\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)). \qquad (2.2)
\end{aligned}$$

Proof. Since $f(x)$ is a monotonic nondecreasing mapping on $[a, b]$, we have

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
= &\left| \int_a^{t_1} (f(x) - f(a))dx + \int_{t_1}^{t_3} (f(x) - f(t_2))dx + \int_{t_3}^{b} (f(x) - f(b))dx \right| \\
= &\left| \left[\int_a^{t_1} (f(x) - f(a))dx + \int_{t_2}^{t_3} (f(x) - f(t_2))dx \right] \right. \\
&\left. - \left[\int_{t_1}^{t_2} (f(t_2) - f(x))dx + \int_{t_3}^{b} (f(b) - f(x))dx \right] \right| \\
\le &\max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&(t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \\
\le &\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).
\end{aligned}$$

Thus (2.1) and (2.2) are proved.

Corollary 1. *Let f be defined as in Theorem 3. Then*

$$\begin{aligned}&\left|\int_a^b f(x)dx - [(x-a)f(a)+(b-x)f(b)]\right| \\ \le&\max\{(b-x)(f(b)-f(x)),(x-a)(f(x)-f(a))\} \\ \le&\max\{x-a,b-x\}\max\{(f(x)-f(a)),(f(b)-f(x))\} \\ \le&\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](f(b)-f(a)).\end{aligned}$$

For $x=(a+b)/2$, we get trapezoid inequality.

Corollary 2. *Let f be defined as in Theorem 3. Then*

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \\ \le&\frac{b-a}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right)-f(a)\right),\left(f(b)-f\left(\frac{a+b}{2}\right)\right)\right\} \qquad (2.3) \\ \le&\frac{1}{2}(b-a)(f(b)-f(a)).\end{aligned}$$

For $t_1=a$, $t_2=x$, $t_3=b$, we get Theorem 1.

3. Application for Special Means.

In this section, we shall give application of Corollary 2. Let us recall the following means.

1. The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\ge 0.$$

2. The geometric mean:

$$G=G(a,b):=\sqrt{ab},\quad a,b\ge 0.$$

3. The harmonic mean:

$$H=H(a,b):=\frac{2}{1/a+1/b},\quad a,b\ge 0.$$

4. The logarthmic mean:

$$L=L(a,b):=\frac{b-a}{\ln b-\ln a},\quad a,b\ge 0, a\ne b;\ If\ a=b,\ then\ L(a,b)=a.$$

5. The identric mean:

$$I=I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},\quad a,b\ge 0,a\ne b;\ If\ a=b,\ then\ I(a,b)=a$$

6. The p-logarthmic mean:

$$L_p = L_p(a,b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, \quad a \neq b;\ If\ a = b,\ then\ L_p(a,b) = a,$$

where $p \neq -1, 0$ and $a, b > 0$.

The following simple relationships are known in the literature

$$H \le G \le L \le I \le A.$$

We are going to use inequality (2.3) in the following equivalent version:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \\ \le&\frac{1}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right) - f(a)\right), \left(f(b) - f\left(\frac{a+b}{2})\right)\right\} \\ \le&\frac{1}{2}(f(b) - f(a)), \end{aligned} \tag{3.1}$$

where $f : [a,b] \to R$ is monotonic nondecreasing on $[a,b]$.

5.1. Mapping $f(x) = x^p$

Consider the mapping $f : [a,b] \subset (0,\infty) \to R, f(x) = x^p, p > 0$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = L_p^p(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^p, b^p),$$

$$f(b) - f(a) = p(b-a)L_{p-1}^{p-1}.$$

Then by (3.1), we get

$$\begin{aligned} \left|L_p^p(a,b) - A(a^p,b^p)\right| \le&\frac{1}{2}\max\left\{\left(\frac{a+b}{2}\right)^p - a^p, b^p - \left(\frac{a+b}{2}\right)^p\right\} \\ =&\frac{1}{2}\left[b^p - \left(\frac{a+b}{2}\right)^p\right] = \frac{1}{2}\left(b^p - a^p\right) - \frac{1}{2}\left(\left(\frac{a+b}{2}\right)^p - a^p\right) \\ \le&\frac{1}{2}p(b-a)L_{p-1}^{p-1} - \frac{p(b-a)a^{p-1}}{4}. \end{aligned} \tag{3.2}$$

Remark 1. The following result was proved in [2].

$$\left|L_p^p(a,b) - A(a^p,b^p)\right| \le \frac{1}{\ }(b-a)L^{p-1}$$

3.2. Mapping $f(x) = -1/x$

Consider the mapping $f : [a, b] \subset (0, \infty) \to R, f(x) = -1/x$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = -L^{-1}(a,b),$$
$$\frac{f(a)+f(b)}{2} = -\frac{A(a,b)}{G^2(a,b)},$$
$$f(b)-f(a) = \frac{b-a}{G^2(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}\left|\frac{A(a,b)}{G^2(a,b)} - L^{-1}(a,b)\right| \le& \frac{1}{2}\max\left\{\frac{1}{a}-\frac{2}{a+b}, \frac{2}{a+b}-\frac{1}{b}\right\}\\ =&\frac{1}{2}\frac{b-a}{a(a+b)} = \frac{1}{2}\frac{b-a}{ab} - \frac{1}{2}\frac{b-a}{b(a+b)}\\ \le&\frac{1}{2}\frac{b-a}{G^2(a,b)} - \frac{1}{2}\frac{b-a}{b(a+b)}.\end{aligned}$$

Thus we get

$$0 \le AL - G^2 \le \frac{1}{2}\frac{b}{a+b}(b-a)L. \tag{3.3}$$

Remark 2. The following result was proved in [2].

$$0 \le AG - G^2 \le \frac{1}{2}(b-a)L.$$

3.3. Mapping $f(x) = \ln x$

Consider the mapping $f : [a, b] \subset (0, \infty) \to R, f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = \ln I(a,b),$$
$$\frac{f(a)+f(b)}{2} = \ln G(a,b),$$
$$f(b)-f(a) = \frac{b-a}{L(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}|\ln I(a,b) - \ln G(a,b)| \le& \frac{1}{2}\max\left\{\ln\frac{a+b}{2} - \ln a, \ln b - \ln\frac{a+b}{2}\right\}\\ =&\frac{1}{2}\ln\frac{a+b}{2a} = \frac{1}{2}\frac{b-a}{L(a,b)} - \frac{1}{2}\ln\frac{2b}{a+b}.\end{aligned}$$

Thus we get

$$1 \le \frac{I}{G} \le \sqrt{\frac{a+b}{2b}}e^{\frac{1}{2}\frac{b-a}{L(a,b)}}. \tag{3.4}$$

Remark 3. The following result was proved in [2].

$$1 \le \frac{I}{G} \le e^{\frac{1}{2}\frac{b-a}{L(a,b)}}.$$

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