NOTE ON MATHIEU'S INEQUALITY

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Abstract. In this note, using the integral expression of Mathieu's series and inequalities for Fourier sine transformation, we establish some new inequalities for Mathieu's series.

1. Introduction

The Mathieu's series is first defined in [7] as

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}, \quad r>0. \tag{1}$$

In 1890, Mathieu [7] conjectured that $S(r)<1/r^2$. In [6], Makai proved

$$\frac{1}{r^2+1/2}<S(r)<\frac{1}{r^2}. \tag{2}$$

The integral expression of Mathieu's series (1) was given in [3, 4] by

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{x\sin(rx)}{\mathrm{e}^x-1}\,\mathrm{d}x. \tag{3}$$

Recently, H. Alzer, J. L. Brenner and O. G. Ruehr in [1] obtained

$$\frac{1}{x^2+1/(2\zeta(3))}<S(x)<\frac{1}{x^2+1/6}, \tag{4}$$

where ζ denots the zeta function.

The study of Mathieu's series and its inequalities has a rich literature, many interesting refinements and extensions of Mathieu's inequality can be found in [1]–[9].

In this paper, using the integral expression (3) of Mathieu's series and inequalities for Fourier sine transformation, we will establish some new inequalities for Mathieu's series.

2. Main Results

We first calculate the integral of Mathieu's series on $(0,+\infty)$.

Proposition 1. *The integral of the Mathieu's series on the infinity interval* $(0,+\infty)$

$$\int_0^{\infty}S(r)\,\mathrm{d}r=\frac{\pi^3}{12}. \tag{5}$$

1991 *Mathematics Subject Classification.* Primary 33E20, 40A30; Secondary 26A48, 26D15.

Key words and phrases. Inequality, integral expression, Mathieu's series, Fourier sine transformation.

The author was supported in part by NSF of Henan Province (no. 004051800), SF for Pure Research of the Education Committee of Henan Province (no. 1999110004), and Doctor Fund of Jiaozuo Institute of Technology, The People's Republic of China.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Proof. It is well-known [11, p. 269 and p. 332] that, for any $\mu > 0$, we have

$$\int_0^\infty \frac{x}{\mathrm{e}^x - 1}\,\mathrm{d}x = \frac{\pi^2}{6}, \qquad \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \int_0^\infty \frac{\sin(\mu x)}{x}\,\mathrm{d}x = \frac{\pi}{2}. \tag{6}$$

Therefore, from formula (3), the integral of Mathieu's series $S(r)$ on the interval $(0, +\infty)$ equals

$$\begin{aligned}\int_0^\infty S(r)\,\mathrm{d}r &= \int_0^\infty \frac{x}{\mathrm{e}^x - 1}\left(\int_0^\infty \frac{\sin(rx)}{r}\,\mathrm{d}r\right)\mathrm{d}x \\ &= \frac{\pi}{2}\int_0^\infty \frac{x}{\mathrm{e}^x - 1}\,\mathrm{d}x \\ &= \frac{\pi^3}{12}.\end{aligned}$$

The proof of Proposition 1 is complete. □

Corollary 1. *The value $\zeta(3)$ of the zeta function ζ at point 3 satisfies*

$$\zeta(3) < \frac{\pi^4}{72}. \tag{7}$$

Proof. Integrating the left hand side of inequalities (4) on the interval $(0, +\infty)$ yields

$$\frac{\pi}{2}\sqrt{2\zeta(3)} = \int_0^\infty \frac{\mathrm{d}x}{x^2 + 1/(2\zeta(3))} < \int_0^\infty S(x)\,\mathrm{d}x = \frac{\pi^3}{12}.$$

This completes the proof of Corollary 1. □

In order to obtain our main result, the following lemma is necessary.

Lemma 1 ([2, pp. 89–90])**.** *If $f \in L([0, \infty))$ with $\lim_{t\to\infty} f(t) = 0$, then*

$$\sum_{k=1}^\infty (-1)^k f(k\pi) < \int_0^\infty f(t)\cos t\,\mathrm{d}t < \sum_{k=0}^\infty (-1)^k f(k\pi), \tag{8}$$

$$\sum_{k=0}^\infty (-1)^k f\left(\left(k + \frac{1}{2}\right)\pi\right) < \int_0^\infty f(t)\sin t\,\mathrm{d}t < f(0) + \sum_{k=0}^\infty (-1)^k f\left(\left(k + \frac{1}{2}\right)\pi\right). \tag{9}$$

Our main result is as follows.

Theorem 1. *If $r > 0$, then*

$$\frac{\pi}{r^3}\sum_{k=0}^\infty \frac{(-1)^k\left(k + \frac{1}{2}\right)}{\exp\left(\left(k + \frac{1}{2}\right)\pi/r\right) - 1} < \sum_{n=1}^\infty \frac{2n}{(n^2 + r^2)^2} < \frac{1}{r^2}\left(1 + \frac{\pi}{r}\sum_{k=0}^\infty \frac{(-1)^k\left(k + \frac{1}{2}\right)}{\exp\left(\left(k + \frac{1}{2}\right)\pi/r\right) - 1}\right). \tag{10}$$

Proof. Since

$$\begin{aligned}S(r) &= \frac{1}{r}\int_0^\infty \frac{x\sin(rx)}{\mathrm{e}^x - 1}\,\mathrm{d}x \\ &= \frac{1}{r^3}\int_0^\infty \frac{t\sin t}{\mathrm{e}^{t/r} - 1}\,\mathrm{d}t,\end{aligned} \tag{11}$$

and, for fixed $r > 0$, let $f(t) = \frac{t}{\mathrm{e}^{t/r} - 1}$, then $f \in L([0, \infty))$,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{t}{\mathrm{e}^{t/r} - 1} = 0, \tag{12}$$

therefore, using inequalities in (9), we have

$$\frac{1}{r^2}\sum_{k=0}^{\infty}\frac{(-1)^k\left(k+\frac{1}{2}\right)\pi/r}{\exp\left(\left(k+\frac{1}{2}\right)\pi/r\right)-1}<S(r)<\frac{1}{r^2}\left(1+\sum_{k=0}^{\infty}\frac{(-1)^k\left(k+\frac{1}{2}\right)\pi/r}{\exp\left(\left(k+\frac{1}{2}\right)\pi/r\right)-1}\right).$$

The proof is complete. □

Open Problem. *Let*

$$S(r,t)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^{t+1}}, \tag{13}$$

where $t>0$ and $r>0$.

Can one get an integral expression of $S(r,t)$ similar to (3)?

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