A STRENTHENED CARLEMAN'S INEQUALITY

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ABSTRACT. In this paper, we obtain a strenthened Carleman's inequality by decreasing its weight coefficient ,and give its order relation in a series of refined Carleman's inequalities.

1. INTRODUCTION

Let $\{a_i\}_{n=1}^{+\infty}$ is a nonnegative sequence such that $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$, we know that the following inequality

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \leq e\sum_{n=1}^{+\infty} a_n \tag{1.1}$$

is called Carleman's inequality. The equality in (1.1) holds if and only if $a_n = 0$, $n = 1, 2, \cdots$, the coefficient e is the optima, for details please refer to [1,2].

For our convenience, we write $\alpha_1 = \frac{1}{\ln 2} - 1$, $\beta_1 = 1 - \frac{2}{e}$, in following.

Recently, in reference [3], we have obtained a series of refined Carleman's inequalities. It is

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \leq e\sum_{n=1}^{+\infty}\left(1-\frac{\beta_1}{n}\right)a_n \tag{1.2}$$

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \leq e\sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\frac{1}{n}\right)^{\alpha_1}} \tag{1.3}$$

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \leq e\sum_{n=1}^{+\infty}\frac{\left(1-\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)^{\alpha}}a_n \tag{1.4}$$

where α, β satisfy $0 \leq \alpha \leq \alpha_1$, $0 \leq \beta \leq \beta_1$, and $e\beta + 2^{1+\alpha} = e$. and, in reference [4], we have obtained their order relations. It is

(1.5)

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \leq e\sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\frac{1}{n}\right)^{\alpha_1}} \leq e\sum_{n=1}^{+\infty}\frac{\left(1-\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)^{\alpha}}a_n \leq e\sum_{n=1}^{+\infty}\left(1-\frac{\beta_1}{n}\right)a_n \leq e\sum_{n=1}^{+\infty}a_n$$

where α, β satisfy $0 \leq \alpha \leq \alpha_1$, $0 \leq \beta \leq \beta_1$, and $e\beta + 2^{1+\alpha} = e$.

In this article, we'll further strengthen the inequality (1.5), and obtain their order relations.

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2. The Strengthened carleman's inequality

In order to give the strengthened Carleman's inequality, first we have

Lemma 1. *For $n = 1, 2, \cdots$. inequality*

$$\left(1+\frac{1}{n}\right)^n < \frac{e}{1+\frac{1}{2n+1}} \tag{2.1}$$

holds.

Proof. Inequality (2.1) is equivalent to

$$\left(1+\frac{1}{2n+1}\right)\left(1+\frac{1}{n}\right)^n < e \tag{2.2}$$

i.e.

$$\ln\left(1+\frac{1}{2n+1}\right) + n\ln\left(1+\frac{1}{n}\right) < 1 \tag{2.3}$$

We have

$$1+\frac{1}{n} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} \tag{2.4}$$

and

$$\ln(1+x) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{x^n}{n}, \qquad -1 < x \le 1 \tag{2.5}$$

$$\ln\frac{1+x}{1-x} = 2\sum_{n=1}^{+\infty}\frac{x^{2n-1}}{2n-1}, \qquad -1 < x < 1 \tag{2.6}$$

Let $x = \frac{1}{2n+1}$, with (2.4), (2.5) and (2.6), we have

$$\ln\left(1+\frac{1}{2n+1}\right) = \sum_{k=1}^{+\infty}(-1)^{k-1}\frac{1}{k(2n+1)^k} \tag{2.7}$$

and

$$\ln(1+\frac{1}{n}) = 2\sum_{k=1}^{+\infty}\frac{1}{(2k-1)(2n+1)^{2k-1}} \tag{2.8}$$

therefore

$$\begin{aligned}
\ln\left(1+\frac{1}{2n+1}\right) + n\ln\left(1+\frac{1}{n}\right) &= \sum_{k=1}^{+\infty}(-1)^{k-1}\frac{1}{k(2n+1)^k} + 2n\sum_{k=1}^{+\infty}\frac{1}{(2k-1)(2n+1)^{2k-1}} \\
&= \left(\frac{1}{2n+1}+\frac{2n}{2n+1}\right) + \left(\frac{2n}{3(2n+1)^3} - \frac{1}{2(2n+1)^2} + \frac{1}{3(2n+1)^3}\right) + \cdots \\
&\quad + \left(\frac{2n}{(2k+1)(2n+1)^{2k+1}} - \frac{1}{2k(2n+1)^{2k}} + \frac{1}{(2k+1)(2n+1)^{2k+1}}\right) + \cdots \\
&= 1 - \sum_{k=1}^{+\infty}\frac{1}{2k(2k+1)(2n+1)^{2k}} < 1
\end{aligned}$$

Inequality (2.3), i.e. (2.2) holds, so inequality (2.1) holds. the proof of lemma 2.1 is complete. □

Theorem 1. *Let* $\{a_i\}_{n=1}^{+\infty}$ *is a nonnegative sequence such that* $0 \le \sum_{n=1}^{+\infty} a_n < +\infty$, *we have inequality*

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{n=1}^{+\infty} \frac{a_n}{1+\frac{1}{2n+1}} \tag{2.9}$$

Proof. Let $c_i > 0$ $(i = 1, 2, \cdots)$, according to arithmetic-geometric mean inequality, we have

$$(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \le \frac{1}{n} \sum_{m=1}^{n} c_m a_m \tag{2.10}$$

Consequently

$$\begin{aligned}\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{+\infty}\left(\frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n}\right)^{1/n} \\ &= \sum_{n=1}^{+\infty}(c_1 c_2 \cdots c_n)^{-1/n}(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \\ &\le \sum_{n=1}^{+\infty}(c_1 c_2 \cdots c_n)^{-1/n}\frac{1}{n}\sum_{m=1}^{n} c_m a_m \\ &= \sum_{m=1}^{+\infty} c_m a_m \sum_{n=m}^{+\infty}\frac{1}{n}(c_1 c_2 \cdots c_n)^{-1/n}\end{aligned}$$

Let $c_m = \frac{(m+1)^m}{m^{m-1}}$ $(m = 1, 2, \cdots, n)$, then $c_1 c_2 \cdots c_n = (n+1)^n$, and

$$\sum_{n=m}^{+\infty} \frac{1}{n}(c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{m} \tag{2.11}$$

Therefore

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \le \sum_{m=1}^{+\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{+\infty} \left(1+\frac{1}{m}\right)^m a_m \tag{2.12}$$

According to lemma 2.1 and substituting for $\left(1+\frac{1}{m}\right)^m$ of inequality (2.12),we have

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \frac{a_m}{1+\frac{1}{2m+1}}$$

So, inequality (2.9) holds. the proof of theorem 2.2 is complete. □

3. The order relations in a series of refined Carleman's inequalities.

In this section, we'll prove that inequality in theorem 2.2 is more exact than the inequalities in reference [4]. We have

Theorem 2. *Let* $\{a_i\}_{n=1}^{+\infty}$ *is a nonnegative sequence such that* $0 \le \sum_{n=1}^{+\infty} a_n < +\infty$, *if*

$$\sum_{n=2}^{+\infty} \left(\frac{1}{\left(1+\frac{1}{n}\right)^{\alpha_1}} - \frac{1}{1+\frac{1}{2n+1}}\right) a_n \ge \left(\frac{3}{4} - \frac{2}{e}\right) a_1 \tag{3.1}$$

we have

$$\sum_{n=1}^{+\infty} \frac{a_n}{1+\frac{1}{2n+1}} \le \sum_{n=1}^{+\infty} \frac{a_n}{\left(1+\frac{1}{n}\right)^{\alpha_1}} \tag{3.2}$$

Proof. First we prove that for $n = 2, 3, \cdots$, the following inequality

$$\left(1+\frac{1}{n}\right)^{\alpha_1} \le 1+\frac{1}{2n+1} \tag{3.3}$$

holds.

Inequality (3.3) is equivalent to

$$1 \le \frac{1+\frac{1}{2n+1}}{\left(1+\frac{1}{n}\right)^{\alpha_1}} \tag{3.4}$$

for $n = 2, 3, \cdots$.

Let

$$f(x) = \frac{2(1+x)}{(2+x)(1+x)^{\alpha_1}} - 1, \quad 0 < x \le 1 \tag{3.5}$$

it is easy to compute that

$$\frac{d}{dx}f(x) = \frac{2}{(2+x)^2(1+x)^{\alpha_1}}\left[(1-2\alpha_1) - \alpha_1 x\right] \tag{3.6}$$

Let

$$h(x) = (1-2\alpha_1) - \alpha_1 x \tag{3.7}$$

it is apparent that $x_0 = \frac{1-2\alpha_1}{\alpha_1}$ is the root of $h(x)$, and $h(x) > 0$, for $0 < x < x_0$; $h(x) < 0$, for $x_0 < x \le 1$. therefore $\frac{d}{dx}f(x) > 0$, for $0 < x < x_0$; $\frac{d}{dx}f(x) < 0$, for $x_0 < x \le 1$. and we know that $f(x)$ is monotone increasing in $(0, x_0)$, and monotone decreasing in $(x_0, 1)$, respectively.

It is apparent that $f(0) = 0$, $f(1) = \frac{8}{3e} - 1 < 0$, and we know that there is only one point x_1 in $(x_0, 1)$ satisfying $f(x_1) = 0$, and we can compute $x_1 \in (0.5, 1)$ due to $f(0.5) = \frac{1.2}{1.5^{\alpha_1}} - 1 > 0$. With these we have $f(x) \ge \min\{f(0), f(x_1)\} = 0$, for $x \in (0, x_1]$. consequently, inequality (3.4), i.e. (3.3) holds.

we can directly compute that $\left(1+\frac{1}{n}\right)^{\alpha_1} > 1 + \frac{1}{2n+1}$, for $n = 1$. with inequality (3.1) we can prove that inequality (3.2) holds. the prove of theorem 3.1 is complete. □

Remark 1. *With theorem 3.1 and inequality (1.5), we can refine Carleman's inequality as*

$$\begin{aligned} \sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} &\le e\sum_{n=1}^{+\infty}\frac{a_n}{1+\frac{1}{2n+1}} \le e\sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\frac{1}{n}\right)^{\alpha_1}} \\ &\le e\sum_{n=1}^{+\infty}\frac{\left(1-\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)^{\alpha}}a_n \le e\sum_{n=1}^{+\infty}\left(1-\frac{\beta_1}{n}\right)a_n \le e\sum_{n=1}^{+\infty}a_n \end{aligned} \tag{3.8}$$

if inequality (3.1) holds. where α, β *satisfy* $0 \le \alpha \le \alpha_1$, $0 \le \beta \le \beta_1$, *and* $e\beta + 2^{1+\alpha} = e$.

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