REFINEMENTS OF CARLEMAN'S INEQUALITY

BAO-QUAN YUAN

ABSTRACT. In this paper, we obtain a series of refined Carleman's Inequalies with Arithmetic-Geometric mean inequality by decreasing their weight coefficient.

1. Introduction

Let $\{a_i\}_{n=1}^{+\infty}$ is a nonnegative sequence such that $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$,then, we have

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \leq e\sum_{n=1}^{+\infty} a_n \tag{1.1}$$

The equality in (1.1) holds if and only if $a_n = 0, n = 1, 2, \cdots$. the coefficient e is optimal

Inequality (1.1) is called Carleman's Inequality, for details please refer to [1, 2]. Though the coefficient e is optimal, we can refine its weight coefficient. In this article we give a series of improved Carleman's inequalities by decreasing the weight coefficient with the arithmetic-geometric mean inequality.

2. The Two Special Cases

In this section, we give two special cases of refined Carleman's inequality .First we prove two lemmas.

Lemma 2.1. *For $m = 1, 2, \cdots$, the following inequality*

$$\left(1+\frac{1}{m}\right)^m \leq e\left(1-\frac{1-2/e}{m}\right) \tag{2.1}$$

holds, where $1-\frac{2}{e} \approx 0.2642411$ is best possible.

Proof. Let

$$\left(1+\frac{1}{m}\right)^m \leq e\left(1-\frac{\beta}{m}\right) \tag{2.2}$$

Then, it is equivalent to

$$\beta \leq m - \frac{m}{e}\left(1+\frac{1}{m}\right)^m,$$

Let $f(x) = \frac{1}{x} - \frac{1}{ex}(1+x)^{\frac{1}{x}} \qquad x \in (0, 1]$

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It's obvious that the function $f(x)$ is a monotone decreasing function on interval $(0,1]$. Consequently, $\beta = f(1) = 1 - \frac{2}{e}$ is the optimal value of satisfying inequality (2.2), So (2.1) holds. The proof of lemma 2.1 follows. ∎

Lemma 2.2. *For $m = 1, 2, \cdots$, the following inequality*

$$\left(1+\frac{1}{m}\right)^m \leq \frac{e}{\left(1+\frac{1}{m}\right)^{\frac{1}{\ln 2}-1}} \tag{2.3}$$

holds, where $\frac{1}{\ln 2} - 1 \approx 0.442695$ is the best possible.

Proof. Let

$$\left(1+\frac{1}{m}\right)^m \leq \frac{e}{\left(1+\frac{1}{m}\right)^{\alpha}} \tag{2.4}$$

It is equivalent to

$$\alpha \leq \frac{1}{\ln\left(1+\frac{1}{m}\right)} - m$$

Let

$$f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x} \qquad x \in (0,1]$$

Because the function $f(x)$ is a monotone decreasing function on interval $(0,1]$. Consequently, $\alpha = f(1) = \frac{1}{\ln 2} - 1$ is the optimal value of satisfying inequality(2.4), So (2.3) holds. The proof of lemma 2.2 follows. ∎

Theorem 2.3. *Let $\{a_i\}_{n=1}^{+\infty}$ is a nonnegative sequence such that $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$, we have*

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \leq e\sum_{m=1}^{+\infty}\left(1-\frac{1-2/e}{m}\right)a_m \tag{2.5}$$

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \leq e\sum_{m=1}^{+\infty}\frac{a_m}{\left(1+\frac{1}{m}\right)^{\frac{1}{\ln 2}-1}}. \tag{2.6}$$

Proof. Let $c_i > 0$ $(i = 1, 2, \cdots)$, according to arithmetic-geometric mean inequality, we have

$$(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{m=1}^{n} c_m a_m$$

Consequently

$$\begin{aligned}
\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{+\infty}\left(\frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n}\right)^{1/n} \\
&= \sum_{n=1}^{+\infty}(c_1 c_2 \cdots c_n)^{-1/n}(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \\
&\leq \sum_{n=1}^{+\infty}(c_1 c_2 \cdots c_n)^{-1/n}\frac{1}{n}\sum_{m=1}^{n} c_m a_m \\
&= \sum_{m=1}^{+\infty} c_m a_m \sum_{n=m}^{+\infty}\frac{1}{n}(c_1 c_2 \cdots c_n)^{-1/n}
\end{aligned}$$

Let $c_m = \frac{(m+1)^m}{m^{m-1}}$ $(m = 1, 2, \cdots, n)$, $c_1 c_2 \cdots c_n = (n+1)^n$, and

$$\sum_{n=m}^{+\infty} \frac{1}{n}(c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{m}$$

Therefore

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \le \sum_{m=1}^{+\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{+\infty} \left(1 + \frac{1}{m}\right)^m a_m \tag{2.7}$$

According to lemma 2.1 and lemma 2.2, and substituting for $\left(1 + \frac{1}{m}\right)^m$ of inequality (2.7),We have

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \left(1 - \frac{1 - 2/e}{m}\right) a_m$$

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \frac{a_m}{\left(1 + \frac{1}{m}\right)^{\frac{1}{\ln 2} - 1}}$$

The proof is complete. ∎

3. A Series of Refined Carleman's Inequalities

In this section we give a series of refined Carleman's inequalities with lemma 3.1. First we have

Lemma 3.1. *For $m = 1, 2, \cdots$,the following inequality*

$$\left(1 + \frac{1}{m}\right)^m \le \frac{e\left(1 - \frac{\beta}{m}\right)}{\left(1 + \frac{1}{m}\right)^{\alpha}} \tag{3.1}$$

holds, where $0 \le \alpha \le \frac{1}{\ln 2} - 1$, $0 \le \beta \le 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$.

Proof. Inequality (3.1) is equivalent to

$$\beta \le m - \frac{m}{e}\left(1 + \frac{1}{m}\right)^{m+\alpha} \tag{3.2}$$

Let

$$f(x) = \frac{1}{x} - \frac{1}{ex}(1 + x)^{\frac{1}{x} + \alpha}, \ x \in (0, 1], \ 0 \le \alpha \le \frac{1}{\ln 2} - 1$$

then $f(x)$ is a monotone decreasing function of x. Consequently, $\beta = f(1) = 1 - \frac{1}{e} 2^{1+\alpha}$ is the optimal value of satisfying inequality (3.2), i.e. $0 \le \beta \le 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$. So (2.3) holds, The proof is complete. ∎

Remark 3.1. *If $\alpha = 0$, then $\beta = 1 - \frac{2}{e}$, and we obtain lemma 1; if $\beta = 0$, then $\alpha = \frac{1}{\ln 2} - 1$, and we obtain lemma 2.*

Similar to theorem 2.3, according to lemma 3.1, we have

Theorem 3.2. *Let $a_n \ge 0 (n = 1, 2, \cdots), 0 \le \sum_{n=1}^{+\infty} a_n < +\infty$,we have*

$$\sum_{n=1}^{+\infty}(a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \frac{\left(1 - \frac{\beta}{m}\right)}{\left(1 + \frac{1}{m}\right)^{\alpha}} a_m$$

where α, β satisfy $0 \le \alpha \le \frac{1}{\ln 2} - 1$, $0 \le \beta \le 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$.

Remark 3.2. *Theorem 2.3 are two special cases of theorem 3.2, if* $\alpha = 0$, $\beta = 1 - \frac{2}{e}$, *and* $\beta = 0$, $\alpha = \frac{1}{\ln 2} - 1$, *we can obtain (2.5) and (2.6) in theorem 2.3 respectively.*

References

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Department of Mathematics ,Jiaozuo Institute of Technology Jiaozuo City Henan Province 454000 People's Republic of China, baoquanyuan@chinaren.com