SOME FURTHER INEQUALITIES FOR UNIVARIATE MOMENTS AND SOME NEW ONES FOR THE COVARIANCE

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. In this paper some further inequalities for univariate moments are given with particular reference to the expectations of the extreme order statistics. In addition, some inequalities are obtained for the covariance of two continuous random variables.

1. Introduction

The expectation of a continuous random variable is given by

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx,$$

where $f(x)$ is the probability density function of the random variable, X. If the interval of definition is finite, (a,b), then this can be further expressed, using integration by parts, as

$$E(X) = b - \int_a^b F(x)\,dx, \tag{1.1}$$

where $F(x)$ is the associated cumulative distribution function. This result has been exploited variously to obtain inequalities involving the expectation and variance, see for example [1], [2], [3].

The aim of this paper is to provide some additional inequalities utilising a generalisation of (1.1) to higher moments, as well as providing some specific results for the extreme order statistics.

In addition, some results are obtained involving the covariance of two random variables using a bivariate generalisation of (1.1) and generalisations of the inequalities of Grüss and Ostrowski.

2. Univariate Results

Denote by M_n the n^{th} moment $\int_a^b x^n f(x)\,dx$ which, using integration by parts, can be expressed as

$$b^n - n\int_a^b x^{n-1} F(x)\,dx. \tag{2.1}$$

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Consider now

$$\left|\frac{1}{b-a}\int_a^b x^{n-1}F(x)\,dx-\frac{1}{(b-a)^2}\int_a^b x^{n-1}dx\int_a^b F(x)\,dx\right|$$

for which various inequalities can be found. Before exploiting some of these, we express the difference in terms of the distributional moments.

$$\frac{1}{(b-a)^2}\int_a^b x^{n-1}dx\int_a^b F(x)\,dx=\frac{(b^n-a^n)}{n(b-a)^2}\{b-E(X)\}.$$

We therefore have:-

$$\begin{aligned}(2.2)\qquad &\left|\frac{1}{b-a}\int_a^b x^{n-1}F(x)\,dx-\frac{(b^n-a^n)}{n(b-a)^2}(b-E(X))\right|\\ =&\left|\frac{b_n}{n(b-a)}-\frac{M_n}{n(b-a)}-\frac{b(b^n-a^n)}{n(b-a)^2}+\frac{M_1(b^n-a^n)}{n(b-a)^2}\right|\\ =&\left|\frac{ab\left(a^{n-1}-b^{n-1}\right)}{n(b-a)^2}-\frac{M_n}{n(b-a)}+\frac{M_1(b^n-a^n)}{n(b-a)^2}\right|\\ =&\frac{1}{n(b-a)^2}\left|a^n(b-M_1)-b^n(a-M_1)-M_n(b-a)\right|.\end{aligned}$$

Now utilising various inequalities, we can obtain a number of results.

Pre-Grüss.

Using an inequality of [4] applied to (2.2), we have

$$\begin{aligned}(2.3)\qquad &\frac{1}{n(b-a)^2}\left|a^n(b-M_1)-b^n(a-M_1)-M_n(b-a)\right|\\ \le&\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b x^{2(n-1)}dx-\left(\frac{1}{(b-a)}\int_a^b x^{n-1}dx\right)^2\right]^{\frac{1}{2}}\\ =&\frac{1}{2}\left[\frac{b^{2n-1}-a^{2n-1}}{(b-a)(2n-1)}-\left(\frac{b^n-a^n}{n(b-a)}\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

The special case when $n=2$ gives:-

$$\begin{aligned}&\left|a^2(b-E(X))-b^2(a-E(X))-(b-a)\left(\sigma^2+(E(X))^2\right)\right|\\ \le&(b-a)^2\left[\frac{b^3-a^3}{3(b-a)}-\left(\frac{b^2-a^2}{2(b-a)}\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

that is,

$$\begin{aligned}&\left|(b-E(X))(E(X)-a)-\sigma^2\right|\\ \le&(b-a)^2\left[\frac{b^2+ab+a^2}{3}-\frac{\left(b^2+2ab+a^2\right)}{4}\right]^{\frac{1}{2}}\\ =&\frac{(b-a)^3}{2\sqrt{3}},\end{aligned}$$

which is Theorem 3 of [3].

Pre-Chebychev.
Using a further result of [4] and (2.2) we can obtain

$$\begin{aligned}&\frac{1}{n(b-a)^2}\left|a^n(b-M_1)-b^n(a-M_1)-M_n(b-a)\right|\\ \leq\ &\frac{(b-a)}{2\sqrt{3}}\|f\|_\infty\left[\frac{1}{(b-a)}\int_a^b x^{2(n-1)}dx-\left(\frac{1}{(b-a)}\int_a^b x^{n-1}dx\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

where $\|f\|_\infty$ is the supremum of the probability density function over (a,b), giving

$$\begin{aligned}&\left|a^n(b-M_1)-b^n(a-M_1)-M_n(b-a)\right|\\ \leq\ &\frac{n(b-a)^3}{2\sqrt{3}}\|f\|_\infty\left[\frac{b^{2n-1}-a^{2n-1}}{(b-a)(2n-1)}-\left(\frac{b^n-a^n}{n(b-a)}\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{2.4}$$

The special case where $n=2$ gives:-

$$\left|(b-E(X))(E(X)-a)-\sigma^2\right|\leq\frac{(b-a)^3}{6}\|f\|_\infty$$

which was obtained by Barnett and Dragomir in [5].

3. Lipschitzian Mappings

If $x^{n-1}F(x)$ is of the Lipschitzian type, then

$$\left|x^{n-1}F(x)-y^{n-1}F(y)\right|\leq L|x-y|,$$

where $L\geq 0$ in which case

$$\left|x^{n-1}F(x)-\frac{1}{b-a}\int_a^b x^{n-1}F(x)\,dx\right|\leq\frac{L}{2}\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right](b-a).$$

(Ostrowski's inequality, [7].)
Now,

$$\left|x^{n-1}F(x)-\frac{1}{b-a}\int_a^b x^{n-1}F(x)\,dx\right|=\left|x^{n-1}F(x)-\left\{\frac{b^n-M_n}{n(b-a)}\right\}\right|$$

and thus we have;-

$$\left|x^{n-1}F(x)-\left\{\frac{b^n-M_n}{n(b-a)}\right\}\right|\leq\frac{L}{2}\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right](b-a).$$

If $n=2$, then

$$\left|xF(x)-\left\{\frac{b^2-M_2}{2(b-a)}\right\}\right|\leq\frac{L}{2}\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right](b-a).$$

Consider now the mapping $F(x)$, $x\in[a,b]$, then the mapping is Lipschitzian if there exists $L>0$ such that

$$|F(x)-F(y)|\leq L|x-y|.$$

Now, if $F(\cdot)$ is a cumulative distribution function, it is monotonic increasing between 0 and 1 over $[a,b]$. It is apparent that there exists $z\in[x,y]$ such that

$$\frac{F(x)-F(y)}{x-y}\leq\left[\frac{dF(x)}{dx}\right]_{x=z}.$$

Thus, if we choose $L=\max\left[\frac{dF(x)}{dx}\right]_{x=z}$ for $z\in[a,b]$, then this implies that $F(x)$ is Lipschitzian, since

$$|F(x)-F(y)|\leq\|f\|_{\infty}|x-y|.$$

Consider similarly the mapping $xF(x)$, it is also monotonic increasing and by the same token, there exists $z\in[x,y]$ such that

$$\left|\frac{xF(x)-yF(y)}{x-y}\right|\leq\left|\left[\frac{d(xF(x))}{dx}\right]_{x=z}\right|.$$

In addition, we have that

$$\left|\left[\frac{d(xF(x))}{dx}\right]_{x=z}\right|=\left|\left[F(x)+x\frac{dF(x)}{dx}\right]_{x=z}\right|$$

and hence

$$\begin{aligned}\left|F(z)+z\left[\frac{dF(x)}{dx}\right]_{x=z}\right| &\leq |F(z)|+\left|z\left[\frac{dF(x)}{dx}\right]_{x=z}\right|\\ &\leq 1+\|f\|_{\infty}\max\{|a|,|b|\}.\end{aligned}$$

Thus, L can be taken to be

$$1+\|f\|_{\infty}\max\{|a|,|b|\}$$

and then

$$|xF(x)-yF(y)|<[1+\|f\|_{\infty}\max\{|a|,|b|\}]|x-y|,$$

and so $xF(x)$ is Lipschitzian.

Similarly it can be shown that $x^{n-1}F(x)$ is Lipschitzian for $n=3,4,\ldots$

Thus,

$$\begin{aligned}&\left|xF(x)-\left\{\frac{b^2-M_2}{2(b-a)}\right\}\right|\\ \leq\ &\frac{1}{2}[1+\|f\|_{\infty}\max\{|a|,|b|\}]\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right](b-a).\end{aligned}$$

For $x=a$ we get

$$\begin{aligned}\left|M_2-b^2\right| &= \left|\sigma^2+\left((E(X))^2\right)-b^2\right|\\ &\leq (b-a)^2[1+\|f\|_{\infty}\max\{|a|,|b|\}]\end{aligned}$$

and for $x=b$ we have

$$\begin{aligned}\left|2b(b-a)-b^2+M_2\right| &= \left|b^2-2ab+M_2\right|\\ &= \left|b(b-2a)+\sigma^2+\left((E(X))^2\right)\right|\\ &\leq (b-a)^2[1+\|f\|_{\infty}\max\{|a|,|b|\}].\end{aligned}$$

4. Distributions of the Maximum, Minimum and Range of a Sample

Consider a continuous random variable X with a non-zero probability density function over a finite interval $[a, b]$. Consider a random sample X_1, X_2,..., X_n. We consider the distribution function of the maximum, minimum and the range of the random sample.

Maximum

Let the cumulative distribution function of the maximum be $G(x)$, the probability density function be $g(x)$, and the corresponding functions for X be $F(x)$ and $f(x)$. Then

$$G(x) = \Pr[\max \leq x] = \Pr[\text{all } X_1, \dots, X_n \leq x].$$

Therefore $G(x) = [F(x)]^n$ and $g(x) = n[F(x)]^{n-1} f(x)$.

Minimum

Let the cumulative distribution function of the minimum and the probability density function be $H(x)$ and $h(x)$ respectively. Therefore

$$\begin{aligned} H(x) &= \Pr[\min \leq x] = 1 - \Pr[\min \geq x] \\ &= 1 - \Pr[\text{all } X_1, \dots, X_n \geq x] = 1 - [1 - F(x)]^n \end{aligned}$$

and

$$h(x) = n[1 - F(x)]^{n-1} f(x).$$

Range

Let the distribution function of the range be $R(x)$ and the probability density function be $r(x)$. Consider

$$\begin{aligned} K(s,t) &= \Pr[\max \leq s, \min \leq t] \\ &= \Pr[\max \leq s] - \Pr[\max \leq s, \min \geq t] \\ &= [F(s)]^n - \Pr\{t \leq \text{all } X_1, \dots, X_n \leq s\} \\ &= [F(s)]^n - [F(s) - F(t)]^n. \end{aligned}$$

Therefore, the joint probability density function of the extreme order statistics can be found by differentiating this with respect to s and t, giving

$$k(s,t) = n(n-1) f(s) f(t) [F(s) - F(t)]^{n-2}, \ s > t$$

which for a random variable defined on $[a, b]$ gives

$$r(x) = n(n-1) \int_{s=a}^{b-x} f(s+x) f(s) [F(s+x) - F(s)]^{n-2} ds,$$

where $0 < x < b - a$.

5. Application of Grüss' Inequality to Positive Integer Powers of a Function

As a preliminary to the proof of Grüss' inequality we can establish the identity:

$$\frac{1}{b-a} \int_a^b g(x) f(x) dx = p + \left(\frac{1}{b-a}\right)^2 \int_a^b f(x) dx \cdot \int_a^b g(x) dx, \tag{5.1}$$

where it can be subsequently shown that

$$|p| \leq \frac{1}{4} [\Gamma - \gamma] [\Phi - \phi],$$

and $\gamma, \Gamma, \phi, \Phi$ are respectively lower and upper bounds of $f(x)$ and $g(x)$.

Applying this same identity to the square of a function, we have

$$\frac{1}{b-a}\int_a^b f^2(x)\,dx = p_1 + \left(\frac{1}{b-a}\right)^2 \left(\int_a^b f(x)\,dx\right)^2 \tag{5.2}$$

Similarly,

$$\frac{1}{b-a}\int_a^b f^3(x)\,dx = p_2 + \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b f^2(x)\,dx$$

and using (5.1), we have

$$\frac{1}{b-a}\int_a^b f^3(x)\,dx = p_2 + \frac{p_1}{b-a}\int_a^b f(x)\,dx + \left(\frac{1}{b-a}\right)^3 \left(\int_a^b f(x)\,dx\right)^3. \tag{5.3}$$

Continuing, we can show that for positive integers n, $n \geq 2$

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n \left(\int_a^b f(x)\,dx\right)^n \\ &= p_{n-1} + p_{n-2}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) + p_{n-3}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \\ &\quad + \cdots + p_1\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^{n-2}, \end{aligned} \tag{5.4}$$

giving:-

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n \left(\int_a^b f(x)\,dx\right)^n\right| \\ &\leq |p_{n-1}| + |p_{n-2}|\left|\frac{1}{b-a}\int_a^b f(x)\,dx\right| + \cdots + |p_1|\left|\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^{n-2}\right|, \end{aligned}$$

where

$$|p_1| \leq \frac{1}{4}(\Gamma-\gamma)^2, \qquad \gamma < f(x) < \Gamma,$$

$$|p_2| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi_1-\phi_1), \quad \phi_1 < f^2(x) < \Phi_1,$$

$$|p_3| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi_2-\phi_2), \quad \phi_2 < f^3(x) < \Phi_2,$$

....

Assuming that $f(x) \geq 0$ and denoting $\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^n$ by λ^n, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n \left(\int_a^b f(x)\,dx\right)^n\right| \qquad (5.5)\\
\leq\ & \frac{1}{4}(\Gamma-\gamma)\left[\Gamma^{n-1}-\gamma^{n-1}\right] + \frac{1}{4}(\Gamma-\gamma)\left[\Gamma^{n-2}-\gamma^{n-2}\right]\lambda \\
&+\frac{1}{4}(\Gamma-\gamma)\left[\Gamma^{n-3}-\gamma^{n-3}\right]\lambda^2 + \cdots + \frac{1}{4}(\Gamma-\gamma)(\Gamma-\gamma)\lambda^{n-2} \\
=\ & \frac{1}{4}(\Gamma-\gamma)\sum_{i=1}^{n-1}\left(\Gamma^{n-i}-\gamma^{n-i}\right)\lambda^{i-1}.
\end{aligned}$$

Now, if f is a p.d.f., the right hand side reduces to

$$\begin{aligned}
&\frac{1}{4}\Gamma\sum_{i=1}^{n-1}\Gamma^{n-i}\left(\frac{1}{b-a}\right)^{i-1} \\
=\ & \frac{1}{4}\Gamma^n\sum_{i=1}^{n-1}\left(\frac{1}{\Gamma(b-a)}\right)^{i-1} = \frac{1}{4}\Gamma^n\left(\frac{1-\left(\frac{1}{\Gamma(b-a)}\right)^{n-1}}{1-\frac{1}{\Gamma(b-a)}}\right) \\
=\ & \frac{\Gamma^n}{4\Gamma^{n-2}(b-a)^{n-2}}\left(\frac{\Gamma^{n-1}(b-a)^{n-1}-1}{\Gamma(b-a)-1}\right) \\
=\ & \frac{\Gamma^2}{4(b-a)^{n-2}}\left(\frac{\Gamma^{n-1}(b-a)^{n-1}-1}{\Gamma(b-a)-1}\right).
\end{aligned}$$

If we now consider this inequality for an associated cumulative distribution function $F(\cdot)$, we have that

$$\lambda = \frac{b-E(X)}{b-a}$$

and the right hand side of (5.5) becomes

$$\frac{1}{4}\sum_{i=1}^{n-1}\left(\frac{b-E(X)}{b-a}\right)^{i-1} = \frac{\left[(b-a)^{n-1}-(b-E(X))^{n-1}\right]}{4(E(X)-a)(b-a)^{n-2}}.$$

Thus, we have the two inequalities:-

$$\left|\frac{1}{b-a}\int_a^b f^n(x)\,dx - \left(\frac{1}{b-a}\right)^n\right| \leq \frac{\Gamma^2}{4(b-a)^{n-2}}\left(\frac{\Gamma^{n-1}(b-a)^{n-1}-1}{\Gamma(b-a)-1}\right) \tag{5.6}$$

and

$$\left|\frac{1}{b-a}\int_a^b F^n(x)\,dx - \left(\frac{b-E(X)}{b-a}\right)^n\right| \leq \frac{\left[(b-a)^{n-1}-(b-E(X))^{n-1}\right]}{4(E(X)-a)(b-a)^{n-2}}. \tag{5.7}$$

Similarly, we can develop an inequality for $1-F(x)$ by suitable substitution in (5.5), that is,

$$\lambda = \frac{E(X)-a}{b-a}$$

which gives

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b (1-F(x))^n \, dx - \left(\frac{1}{b-a} \int_a^b (1-F(x)) \, dx \right)^n \right| \\ = & \left| \frac{1}{b-a} \int_a^b (1-F(x))^n \, dx - \left(\frac{E(X)-a}{b-a} \right)^n \right| \\ \leq & \frac{\left[(b-a)^{n-1} - (E(X)-a)^{n-1} \right]}{4(b-a)^{n-2}(b-E(X))}. \end{aligned} \tag{5.8}$$

6. Inequalities for the Expectation of the Extreme Order Statistics

As the p.d.f. of the maximum is

$$g(x) = n[F(x)]^{n-1} f(x),$$

then

$$E[X_{\max}] = n \int_a^b x [F(x)]^{n-1} f(x) \, dx.$$

Integrating by parts gives

$$\begin{aligned} E[X_{\max}] &= n \left[\left[\frac{x}{n} [F(x)]^n \right]_a^b - \frac{1}{n} \int_a^b (F(x))^n \, dx \right] \\ &= b - \int_a^b F^n(x) \, dx \end{aligned}$$

giving, from (5.7)

$$\left| \frac{b - E[X_{\max}]}{b-a} - \left(\frac{b-E(X)}{b-a} \right)^n \right| \leq \frac{\left[(b-a)^{n-1} - (b-E(X))^{n-1} \right]}{4(E(X)-a)(b-a)^{n-2}} \tag{6.1}$$

and when $E(X) = \frac{a+b}{2}$, we have:-

$$\left| \frac{b - E[X_{\max}]}{b-a} - \frac{1}{2^n} \right| \leq \left(\frac{2^{n-1}-1}{2^n} \right). \tag{6.2}$$

Consider now $E(X_{\min}) = n \int_{x=a}^b x f(x) [1-F(x)]^{n-1} \, dx$. Integration by parts gives:-

$$E[X_{\min}] = n \left[\left[-\frac{x}{n} [1-F(x)]^n \right]_a^b + \frac{1}{n} \int_a^b (1-F(x))^n \, dx \right]$$

and so

$$E(X_{\min}) = a + \int_a^b (1-F(x))^n \, dx.$$

Utilising (5.8) we have

$$\left| \frac{E(X_{\min}) - a}{b-a} - \left(\frac{E(X)-a}{b-a} \right)^n \right| \leq \frac{\left[(b-a)^{n-1} - (E(X)-a)^{n-1} \right]}{4(b-a)^{n-2}(b-E(X))} \tag{6.3}$$

and when $E(X) = \frac{a+b}{2}$, we have:-

$$\left| \frac{E(X_{\min}) - a}{b-a} - \frac{1}{2^n} \right| \leq \left(\frac{2^{n-1}-1}{2^n} \right). \tag{6.4}$$

7. Applications to the Beta Distribution

The Beta probability density function is given by

$$\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},\ 0<x<1,\ \alpha,\beta>0,$$

where

$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx.$$

Clearly,

$$\begin{aligned}E(X) &= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx=\frac{1}{B(\alpha,\beta)}B(\alpha+1,\beta)\\ &= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)}=\frac{\alpha}{(\alpha+\beta)}.\end{aligned}$$

Substituting $a=0$, $b=1$ and letting 'Γ'$\equiv m$, from (5.6) we obtain

$$\left|\int_0^1 f^n(x)\,dx-1\right|\le\frac{m^2\left(1-m^{n-1}\right)}{4(1-m)}$$

and further,

$$\begin{aligned}&\left|\frac{1}{B^n(\alpha,\beta)}\int_0^1 x^{n(\alpha-1)}(1-x)^{n(\beta-1)}\,dx-1\right|\\ =&\left|\frac{B(n(\alpha-1)+1,n(\beta-1)+1)}{B^n(\alpha,\beta)}-1\right|\le\frac{m^2\left(1-m^{n-1}\right)}{4(1-m)}\end{aligned}$$

and m is the value of x for which $f'(x)=0$, that is

$$\begin{aligned}&(1-x)^{\beta-1}(\alpha-1)x^{\alpha-2}+x^{\alpha-1}(\beta-1)(1-x)^{\beta-2}(-1)=0.\\ &x^{\alpha-2}(1-x)^{\beta-2}\{(\alpha-1)(1-x)-x(\beta-1)\},\\ \text{i.e. } m =&\ \frac{\alpha-1}{\alpha+\beta-2},\quad \alpha,\beta>1.\end{aligned}$$

We then have the inequality:-

$$\begin{aligned}&\left|\frac{B(n(\alpha-1)+1,n(\beta-1)+1)}{B^n(\alpha,\beta)}-1\right|\\ \le&\ \frac{(\alpha-1)^2}{4(\alpha+\beta-2)(\beta-1)}\left\{1-\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{n-1}\right\}.\end{aligned}$$

When $\alpha=\beta$, the right hand side becomes $\frac{1}{8}\left(1-\frac{1}{2^{n-1}}\right)$.

Consider now (6.1) when $f(x)$ is the p.d.f. of a Beta distribution. This gives:-

$$\begin{aligned} & \left|1-E\left(X_{\max}\right)-\left(1-\frac{\alpha}{\alpha+\beta}\right)^n\right| \\ = & \left|1-\left(\frac{\beta}{\alpha+\beta}\right)^n-E\left(X_{\max}\right)\right| \leq \frac{\left[1-\left(1-\frac{\alpha}{\alpha+\beta}\right)^{n-1}\right]}{4\left(\frac{\alpha}{\alpha+\beta}\right)} \\ = & \frac{(\alpha+\beta)^{n-1}-\beta^{n-1}}{4\alpha(\alpha+\beta)^{n-2}} \end{aligned}$$

and when $\alpha=\beta$, this becomes:-

$$\left|1-\left(\frac{1}{2}\right)^n-E\left(X_{\max}\right)\right| \leq \frac{1-\left(\frac{1}{2}\right)^{n-1}}{2}$$

and from (6.3)

$$\left|E\left(X_{\min}\right)-\left(\frac{\alpha}{\alpha+\beta}\right)^n\right| \leq \frac{\left[1-\left(\frac{\alpha}{\alpha+\beta}\right)^{n-1}\right]}{4\left(1-\frac{\alpha}{\alpha+\beta}\right)}=\frac{(\alpha+\beta)^{n-1}-\alpha^{n-1}}{4\beta(\alpha+\beta)^{n-2}}$$

and when $\alpha=\beta$

$$\left|E\left(X_{\min}\right)-\frac{1}{2^n}\right| \leq \frac{1-\left(\frac{1}{2}\right)^{n-1}}{2}.$$

From these we can obtain further $E\left(X_{\min}\right) \leq \frac{1}{2}$ and $\frac{1}{2} \leq E\left(X_{\max}\right) \leq \frac{3}{2}-\left(\frac{1}{2}\right)^{n-1}$.

8. Some Bounds for Joint Moments and Probabilities Using Ostrowski Type Inequalities for Double Integrals

Theorem 1. *Let X, Y be two continuous random variables $x \in [a,b]$, $y \in [c,d]$ with probability density functions $f_1(\cdot)$ and $f_2(\cdot)$ respectively and with joint probability density function $f(\cdot,\cdot)$ with associated cumulative distribution functions $F_1(\cdot)$, $F_2(\cdot)$ and $F(\cdot,\cdot)$. Then*

$$E(XY)=bE(Y)+dE(X)-bd+\int_{s=a}^{b}\int_{t=c}^{d}F(s,t)\,dsdt. \tag{8.1}$$

This is a generalisation of the result

$$E(X)=b-\int_a^b F(s)\,ds$$

and is equivalent to:-

$$E(XY)=bd-d\int_a^b F_1(s)\,ds-b\int_c^d F_2(t)\,dt+\int_{s=a}^{b}\int_{t=c}^{d}F(s,t)\,dsdt. \tag{8.2}$$

Proof.

$$E(XY)=\int_{t=c}^{d} t\left\{\int_a^b sf(s,t)\,ds\right\}dt$$

and

$$\begin{aligned}\int_a^b sf(s,t)\,ds &= \left[s\int_{u=a}^{s} f(u,t)\,du\right]_{s=a}^{b} - \int_{s=a}^{b}\left(\int_{u=a}^{s} f(u,t)\,du\right)ds\\ &= bf_2(t) - \int_{s=a}^{b}\left(\int_{u=a}^{s} f(u,t)\,du\right)ds,\end{aligned}$$

so

$$\begin{aligned}E(XY) &= b\int_c^d tf_2(t)\,dt - \int_{t=c}^{d} t\left[\int_{s=a}^{b}\left(\int_{u=a}^{s} f(u,t)\,du\right)ds\right]dt\\ &= bE(Y) - \int_{s=a}^{b}\left(\int_{u=a}^{s}\left(\int_{t=c}^{d} tf(u,t)\,dt\right)du\right)ds\end{aligned}$$

Now

$$\begin{aligned}\int_{t=c}^{d} tf(u,t)\,dt &= \left[t\int_{v=c}^{t} f(u,v)\,dv\right]_{t=c}^{d} - \int_{t=c}^{d}\left(\int_{v=c}^{t} f(u,v)\,dv\right)dt\\ &= df_1(u) - \int_{t=c}^{d}\left(\int_{v=c}^{t} f(u,v)\,dv\right)dt,\end{aligned}$$

$$\begin{aligned}E(XY) &= bE(Y) - \int_{s=a}^{b}\left(\int_{u=a}^{s}\left\{df_1(u) - \int_{t=c}^{d}\left(\int_{v=c}^{t} f(u,v)\,dv\right)dt\right\}du\right)ds,\\ &= bE(Y) - d\int_{s=a}^{b} F_1(s)\,ds\\ &\quad + \int_{t=c}^{d}\left(\int_{s=a}^{b}\left(\int_{v=c}^{t}\left(\int_{u=a}^{s} f(u,v)\,du\right)dv\right)ds\right)dt\\ &= bE(Y) + dE(X) - bd + \int_{s=a}^{b}\int_{t=c}^{d} F(s,t)\,dsdt\end{aligned}$$

and, equivalently,

$$E(XY) = bd - d\int_a^b F_1(s)\,ds - b\int_c^d F_2(t)\,dt + \int_a^b\int_c^d F(s,t)\,dsdt.$$

□

In [6] Barnett and Dragomir proved the following theorem.

Theorem 2. *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,*

$$\left\|f''_{s,t}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| < \infty$$

then we have the inequality:

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(s,t)\,dsdt\right. \\
&\left.-\left[(b-a)\int_c^d f(x,t)\,dt+(d-c)\int_a^b f(s,y)\,ds-(d-c)(b-a)f(x,y)\right]\right| \\
\leq\;&\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] \\
&\times\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_\infty
\end{aligned} \tag{8.3}$$

for all $(x,y)\in[a,b]\times[c,d]$.

If we apply this taking $f(\cdot,\cdot)$ to be a joint cumulative distribution function $F(\cdot,\cdot)$ with $x=b$, $y=d$ we obtain

$$\begin{aligned}
&\left|\int_a^b\int_c^d F(s,t)\,dsdt\right. \\
&\left.-(b-a)\int_c^d F(b,t)\,dt-(d-c)\int_a^b F(s,d)\,ds+(d-c)(b-a)\right| \\
\leq\;&\frac{1}{4}(b-a)^2(d-c)^2\left\|F''_{s,t}\right\|_\infty
\end{aligned}$$

That is

$$\begin{aligned}
&\left|\int_a^b\int_c^d F(s,t)\,dsdt\right. \\
&\left.-(b-a)\int_c^d F_2(t)\,dt-(d-c)\int_a^b F_1(s)\,ds+(d-c)(b-a)\right| \\
\leq\;&\frac{1}{4}(b-a)^2(d-c)^2\left\|F''_{s,t}\right\|_\infty .
\end{aligned}$$

Using (8.2), this gives

$$\begin{aligned}
&\left|E(XY)+a\int_c^d F_2(t)\,dt+c\int_a^b F_1(s)\,ds-ad-bc+ac\right| \\
=\;&|E(XY)+aE(Y)-cE(X)+ac|\leq\frac{1}{4}(b-a)^2(d-c)^2\left\|F''_{s,t}\right\|_\infty \\
=\;&\frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty ,
\end{aligned}$$

providing bounds for $E(XY)$ in terms of $E(X)$ and $E(Y)$.

Since $Cov(X,Y)=E(XY)-E(X)\cdot E(Y)$, we can write the left hand side alternatively as:-

$$Cov(X,Y)+[c-E(Y)][a-E(X)].$$

We can similarly extract other bounds from (8.3) in the situations where

(i) $x=b,\ y=c$,
(ii) $x=a,\ y=d$, and

(iii) $x=a,\ y=c$

giving respectively

$$\begin{aligned}
\left|E(XY)-dE(X)-aE(Y)+ad\right| &\leq \frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty,\\
\left|E(XY)-cE(X)-bE(Y)+bc\right| &\leq \frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty
\end{aligned}$$

and

$$\left|E(XY)-dE(X)-bE(Y)+bd\right| \leq \frac{1}{4}(b-a)^2(d-c)^2\|f\|_\infty.$$

We can use the results of [8] by Dragomir, Cerone, Barnett and Roumeliotis to obtain further inequalities relating the first single and joint moments as well as some involving joint probabilities.

In [8], bounds were obtained for:-

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)\,dsdt - f(x,y)\right|$$

namely $M_1(x)+M_2(y)+M_3(x,y)$ where these are as defined in [8]. For one particular case we have

$$\begin{aligned}
M_1(x) &= \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f(s,t)}{\partial t}\right\|_1,\\
M_2(y) &= \frac{\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f(s,t)}{\partial s}\right\|_1
\end{aligned}$$

and

$$M_3(x,y)=\frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right\|_1.$$

It follows then that if we choose f to be the joint cumulative distribution function, $F(x,y)$, we can generate the following inequalities

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(s,t)\,dsdt\right| \leq M_1(a)+M_2(c)+M_3(a,c),$$

and

$$\left|\int_a^b\int_c^d F(s,t)\,dsdt - \Pr\{X\leq x, Y\leq y\}\right| \leq M_1(x)+M_2(y)+M_3(x,y).$$

The first of these simplifies to give:-

$$\left|E(XY)-bE(Y)-dE(X)+bd\right| \leq (b-a)+(d-c)+(d-c)(b-a).$$

9. Further Inequalities for the Covariance Using Grüss Inequality

Consider functions $f_1(\cdot)$, $f_2(\cdot)$ and $f(\cdot,\cdot)$ where f_1 is integrable over $[a,b]$, f_2 is integrable over $[c,d]$ and $f(x,y)$ is integrable for $x\in[a,b]$ and $y\in[c,d]$. Consider the integral

$$\int_a^b\int_c^d f_1(s)f_2(t)f(s,t)\,dsdt = \int_{s=a}^b f_1(s)\left(\int_{t=c}^d f_2(t)f(s,t)\,dt\right)ds.$$

We have:-

$$\begin{aligned} & \frac{1}{d-c}\int_c^d f_2(t) f(s,t)\,dt - \left(\frac{1}{d-c}\right)^2 \int_c^d f_2(t)\,dt \int_{t=c}^d f(s,t)\,dt \\ = \;& p_1(s) \quad \text{(say)}. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_a^b \int_c^d f_1(s) f_2(t) f(s,t)\,dsdt \\ = \;& \int_a^b f_1(s)\left\{p_1(s)(d-c) + \frac{1}{d-c}\int_c^d f_2(t)\,dt \int_c^d f(s,t)\,dt\right\} ds \\ = \;& (d-c)\int_a^b p_1(s) f_1(s)\,ds + \frac{1}{d-c}\int_c^d f_2(t)\,dt \int_c^d \left(\int_a^b f_1(s) f(s,t)\,ds\right) dt \end{aligned}$$

and where

$$\frac{1}{b-a}\int_a^b f_1(s)\cdot f(s,t)\,ds - \frac{1}{(b-a)^2}\int_a^b f_1(s)\,ds \int_a^b f(s,t)\,ds = p_2(t).$$

Therefore

$$\begin{aligned} & \int_a^b \int_c^d f_1(s) f_2(t) f(s,t)\,dsdt \\ = \;& (d-c)\int_a^b p_1(s) f_1(s)\,ds + \frac{1}{(d-c)}\int_c^d f_2(t)\,dt \\ & \times \int_c^d \left[(b-a) p_2(t) + \frac{1}{b-a}\int_a^b f_1(s)\,ds \int_a^b f(s,t)\,ds\right] dt \\ = \;& (d-c)\int_a^b p_1(s) f_1(s)\,ds + \frac{b-a}{(d-c)}\int_c^d f_2(t)\,dt \int_c^d p_2(t)\,dt \\ & + \frac{1}{(b-a)(d-c)}\int_c^d f_2(t)\,dt \int_a^b f_1(s)\,ds \int_a^b \int_c^d f(s,t)\,dsdt \end{aligned}$$

and thus

$$\begin{aligned} & \left| \int_a^b \int_c^d f_1(s) f_2(t) f(s,t)\,dsdt \right. \\ & \left. - \frac{1}{(b-a)(d-c)}\int_c^d f_2(t)\,dt \int_a^b f_1(s)\,ds \int_a^b \int_c^d f(s,t)\,dsdt \right| \\ \leq \;& (d-c)\|p_1\|_\infty \int_a^b |f_1(s)|\,ds + (b-a)\|p_2\|_\infty \int_c^d |f_2(t)|\,dt. \end{aligned} \tag{9.1}$$

Case 1. *Now, if $f_1(s) = s$ and $f_2(t) = t$, and $f(\cdot,\cdot)$ is a joint probability density function, the left hand side becomes*

$$\begin{aligned} & \left| E(XY) - \frac{1}{4(d-c)(b-a)}\left(d^2-c^2\right)\left(b^2-a^2\right) \right| \\ = \;& \left| E(XY) - \frac{1}{4}(b+a)(d+c) \right|. \end{aligned}$$

$$
\begin{aligned}
|p_1(s)| &\leq \frac{1}{2}\|f\|_\infty \left[\frac{\left(d^3-c^3\right)}{3(d-c)} - \left(\frac{1}{d-c}\int_c^d t dt\right)^2\right]^{\frac{1}{2}} \quad \text{see [4]} \\
&= \frac{1}{2}\|f\|_\infty \left[\frac{(d-c)\left(d^2+dc+c^2\right)}{3(d-c)} - \left(\frac{\left(d^2-c^2\right)}{2(d-c)}\right)^2\right]^{\frac{1}{2}} \\
&= \frac{1}{2}\|f\|_\infty \left[\frac{d^2+dc+c^2}{3} - \frac{(d+c)^2}{4}\right]^{\frac{1}{2}} \\
&= \frac{1}{2}\|f\|_\infty \times \frac{1}{2\sqrt{3}}\left[d^2-2dc+c^2\right]^{\frac{1}{2}} = \frac{1}{4\sqrt{3}}\|f\|_\infty (d-c).
\end{aligned}
$$

Similarly,

$$p_2 \leq \frac{a^2-b^2}{2},\ a<0,\ b<0$$

Now

$$\int_a^b |f_1(s)|\, ds = \frac{a^2+b^2}{2},\ a<0,\ b>0$$

and similarly,

$$\int_c^d |f_2(t)|\, dt \leq \frac{b^2-a^2}{2},\ a>0,\ b>0.$$

Thus, we then have for $a<0$, $b>0$, $c<0$, $d>0$:-

$$
\begin{aligned}
&\left|E(XY) - \frac{1}{4}(b+a)(d+c)\right| \\
\leq\ & \frac{1}{4\sqrt{3}}\|f\|_\infty \frac{(d-c)^2\left(a^2+b^2\right)}{2} + \frac{1}{4\sqrt{3}}\|f\|_\infty \frac{(b-a)^2\left(c^2+d^2\right)}{2} \\
=\ & \frac{1}{8\sqrt{3}}\left[(d-c)^2\left(a^2+b^2\right) + (b-a)^2\left(c^2+d^2\right)\right]\|f\|_\infty \\
=\ & \frac{1}{4\sqrt{3}}\left[\left(a^2+b^2\right)\left(c^2+d^2\right) + (ac+db)(ad+bc)\right]\|f\|_\infty .
\end{aligned}
$$

Case 2. *If $f_1(s)=s$ and $f_2(t)=1$, and $f(\cdot,\cdot)=t\phi(s,t)$ where $\phi(\cdot,\cdot)$ is a joint probability density function, then the left hand side is:-*

$$
\begin{aligned}
&\left|\int_a^b\int_c^d st\phi(s,t)\, dsdt - \frac{(d-c)\left(b^2-a^2\right)}{2(d-c)(b-a)}E(Y)\right| \\
=\ & \left|E(XY) - \frac{1}{2}(a+b)E(Y)\right|.
\end{aligned}
$$

p_2 is as above and

$$p_1 \leq \frac{1}{2}\|f\|_\infty \left[\frac{1}{d-c}\int_c^d dt - \left(\frac{1}{d-c}\int_c^d dt\right)^2\right]^{\frac{1}{2}} = 0$$

and hence

$$\left|E(XY) - \frac{1}{2}(a+b)E(Y)\right| \leq \frac{(b-a)^2\left(c^2+d^2\right)}{8\sqrt{3}}\|f\|_\infty$$

when $a<0, b>0, c<0, d>0.$

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: `neil@matilda.vu.edu.au`
URL: `http://sci.vu.edu.au/staff/neilb.html`

E-mail address: `sever@matilda.vu.edu.au`
URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`