# AN INEQUALITY ABOUT EXPONENT $e$

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Abstract. An inequality about exponent $e$ is given and proved, with this inequality we further strengthen Carleman's inequality.

## 1. Introduction

It is common knowledge that the sequence $\left(1+\frac{1}{n}\right)^n$ increases and converges to $e \approx 2.718281828\cdots$ as $n$ approaches infinity. In other words, the inequality

$$\left(1+\frac{1}{n}\right)^n < e \tag{1.1}$$

holds for $n = 1, 2, \cdots$, and $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$. We would like to know whether there are more accurate inequalities than (1.1). In fact, we have obtained an inequality in reference [1], namely

$$\left(1+\frac{1}{2n+1}\right)\left(1+\frac{1}{n}\right)^n < e \tag{1.2}$$

for $n = 1, 2, \cdots$.

With inequality (1.2) we can easily refine Carleman's inequality, for details please refer to [1], for Carleman's inequality please refer to [2] and [3]. In this paper, we first obtain an inequality about $e$, then we refine Carleman's inequality.

## 2. The Inequality About $e$

In this section we obtain an inequality about $e$ by decreasing the sequence weight coefficient $\left(1+\frac{1}{2n+b}\right)$ of sequence $\left(1+\frac{1}{n}\right)^n$.

**Theorem 1.** *For $n = 1, 2, \cdots$, the following inequality*

$$\left(1+\frac{1}{2n+b}\right)\left(1+\frac{1}{n}\right)^n < e \tag{2.1}$$

*holds, where $b \geq \frac{5}{6}$.*

*Proof.* Denote

$$f(x) = \ln\left(1+\frac{1}{2x+b}\right) + x\ln\left(1+\frac{1}{x}\right) - 1. \tag{2.2}$$

Inequality (2.1) is equivalent to

$$f(x) < 0. \tag{2.3}$$

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It is obvious that

$$\lim_{x\to+\infty} f(x) = 0. \tag{2.4}$$

Simple calculation yields that

$$\frac{d}{dx}f(x) = \frac{2}{2x+b+1} - \frac{2}{2x+b} + \ln\left(1+\frac{1}{n}\right) + \frac{x}{x+1} - 1. \tag{2.5}$$

In addition, we have

$$\lim_{x\to+\infty} \frac{d}{dx}f(x) = 0 \tag{2.6}$$

and

$$\frac{d^2}{dx^2}f(x) = -\frac{(24b-20)\,x^3 + \left(24b^2+8b-20\right)x^2 + \left(8b^3+12b^2-4b-4\right)x + b^4 + 2b^3 + b^2}{x\,(1+x)^2\,(2x+b)^2\,(2x+b+1)^2}. \tag{2.7}$$

Since it is obvious that

$$24b - 20 \geq 0 \quad \text{if } b \geq \frac{5}{6} \tag{2.8}$$

$$24b^2 + 8b - 20 \geq 0 \quad \text{if } b \geq \frac{\sqrt{31}-1}{6} \tag{2.9}$$

$$8b^3 + 12b^2 - 4b - 4 \geq 0 \quad \text{if } b \geq \frac{\sqrt{5}-1}{2}, \tag{2.10}$$

we obtain

$$\frac{d^2}{dx^2}f(x) < 0 \quad \text{if } b \geq \max\left(\frac{5}{6}, \frac{\sqrt{31}-1}{6}, \frac{\sqrt{5}-1}{2}\right) = \frac{5}{6} \tag{2.11}$$

and we know that $\frac{d}{dx}f(x)$ is a monotone decreasing function.Using (2.6) we know

$$\frac{d}{dx}f(x) > 0 \tag{2.12}$$

for $x > 0$.

So $f(x)$ is a monotone increasing function. Using (2.4) we obtain

$$f(x) < 0 \tag{2.13}$$

for $x > 0$.

As a result, we prove that (2.1) holds and the proof of Theorem 1 is thus complete. □

**Remark 1.** *In Theorem 1 if we let $b = 1$, we can obtain inequality (1.2).*

## 3. A Series of Strengthened Carleman's Inequalities

In this section we give a series of strengthened Carleman's inequalities using the result of Theorem 1.

**Theorem 2.** *Let* $\{a_i\}_{n=1}^{+\infty}$ *be a nonnegative sequence such that* $0 \le \sum_{n=1}^{+\infty} a_n < +\infty$, *we have*

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{n=1}^{+\infty} \frac{a_n}{1 + \frac{1}{2n+b}}, \tag{3.1}$$

*where* $b \ge \frac{5}{6}$.

*Proof.* Let $c_i > 0$ $(i = 1, 2, \cdots)$, according to arithmetic-geometric mean inequality, we have

$$(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \le \frac{1}{n} \sum_{m=1}^{n} c_m a_m. \tag{3.2}$$

Consequently,

$$\begin{aligned} \sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{+\infty} \left( \frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n} \right)^{1/n} \\ &= \sum_{n=1}^{+\infty} (c_1 c_2 \cdots c_n)^{-1/n} (c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \\ &\le \sum_{n=1}^{+\infty} (c_1 c_2 \cdots c_n)^{-1/n} \frac{1}{n} \sum_{m=1}^{n} c_m a_m \\ &= \sum_{m=1}^{+\infty} c_m a_m \sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n}. \end{aligned} \tag{3.3}$$

Let $c_m = \frac{(m+1)^m}{m^{m-1}}$ $(m = 1, 2, \cdots, n)$, then $c_1 c_2 \cdots c_n = (n+1)^n$, and

$$\sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{m}. \tag{3.4}$$

Therefore,

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{m=1}^{+\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{+\infty} \left(1 + \frac{1}{m}\right)^m a_m \tag{3.5}$$

According to Theorem 1 and substituting for $\left(1 + \frac{1}{m}\right)^m$ of inequality (3.5),we have

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \frac{a_m}{1 + \frac{1}{2m+b}} \tag{3.6}$$

Thus, inequality (3.1) holds and the proof of Theorem 2 is thus complete. □

**Remark 2.** *In Theorem 2, if we let* $b = 1$, *we obtain*

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \frac{a_m}{1 + \frac{1}{2m+1}}, \tag{3.7}$$

*which is the inequality given in* [1].

**Remark 3.** *In Theorem 2, if we let $b = \frac{5}{6}$, we obtain the inequality*

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{a_m}{1 + \frac{1}{2m+5/6}} \tag{3.8}$$

*which is the best one among the inequalities given in Theorem 2, and we obtain the relationships*

(3.9)

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{a_m}{1 + \frac{1}{2m+5/6}} < e \sum_{m=1}^{+\infty} \frac{a_m}{1 + \frac{1}{2m+b}} < e \sum_{m=1}^{+\infty} \frac{a_m}{1 + \frac{1}{2m+1}},$$

*where* $\frac{5}{6} < b < 1$.

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## References

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