ABOUT SEIFFERT'S MEAN

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ABSTRACT. In this paper, a number of inequalities are obtained using Seiffert's mean.

1. Results

In 1995, H. J. Seiffert (see [3]) introduced the following mean

$$M(x,y)=\begin{cases}\frac{x-y}{2\arcsin\left(\frac{x-y}{x+y}\right)} & \text{if} \quad x\neq y,\\ x & \text{if} \quad x=y.\end{cases}$$

In this paper we prove some inequalities for this mean.
Let $A(x,y)=\frac{x+y}{2}$, $G(x,y)=\sqrt{xy}$, and $L(x,y)=\frac{x-y}{\ln x-\ln y}$.

Theorem 1. *The following inequalities hold:*

(1) $$M^2\geq A\cdot L;$$

(2) $$\frac{M}{A}\geq\frac{\pi^2M^2-A^2+G^2}{\pi^2M^2+A^2-G^2};$$

(3) $$\frac{3G}{A+2G}\leq\frac{M}{A}\leq\frac{3A}{4A-G};$$

(4) $$\frac{1}{\sqrt{M^2+A^2-G^2}}\leq\frac{1}{A}\leq\frac{1}{M};$$

(5) $$2AG\leq 2AM+GM;$$

(6) $$GA^2\leq M^3;$$

(7) $$\pi M\geq 3G;$$

(8) $$G\left(A^2-G^2\right)\left(3M^2-A^2+G^2\right)\leq A^2M^3;$$

(9) $$\sin\frac{G}{A}\leq\cos\frac{\sqrt{A^2-G^2}}{A};$$

(10) $$\sin\frac{\sqrt{A^2-G^2}}{A}\leq\cos\frac{G}{A};$$

(11) $$G^2\leq AM;$$

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$$1-\frac{G^2}{A^2}\le\sin\left(\frac{A^2-G^2}{M^2}\right); \tag{12}$$

$$M^2\ge AG; \tag{13}$$

$$\frac{1}{A}+\frac{1}{G}\ge\frac{2}{M}; \tag{14}$$

$$\frac{2}{A}+\frac{1}{G}\ge\frac{3}{M}; \tag{15}$$

$$\frac{1}{M}\le\frac{2}{A+G}\le\frac{2}{M}; \tag{16}$$

$$\frac{1}{G}-\frac{1}{M}\ge\frac{\sqrt{A^2-G^2}}{3GM}\ge\frac{A^2-G^2}{3M^3}; \tag{17}$$

$$\ln^2\frac{G}{A}\le\frac{A^2-G^2}{M}\left(\frac{1}{G}-\frac{1}{M}\right); \tag{18}$$

$$\frac{A-G}{M}+\ln\frac{G}{A}\le 0; \tag{19}$$

$$\frac{nM\sqrt{A^2-G^2}}{M^2+(n+1)^2\left(A^2-G^2\right)}\le tg\left(\ln\sqrt{n+1}\right),\quad \textit{where } n\in\mathbb{N}; \tag{20}$$

$$\left|\left(\frac{\sqrt{A^2-G^2}}{G}\right)^n-\left(\frac{G}{\sqrt{A^2-G^2}}\right)^n\right|\ge 2n\left|1-\frac{2G^2}{A^2}\right|,\quad \textit{where } n\in\mathbb{N}; \tag{21}$$

$$\begin{aligned}&\left(\sqrt{A^2-G^2}+aG\right)\left(\sqrt{A^2-G^2}+bG\right)\\ \le\quad & A^2\left(1+\left(\frac{a+b}{2}\right)^2\right)\ \textit{for}\ \ a,b\in\mathbb{R};\end{aligned} \tag{22}$$

$$\left(1+\frac{aA}{\sqrt{A^2-G^2}}\right)\left(1+\frac{2A}{G}\right)\ge\left(1+\sqrt{2ab}\right)^2\quad \textit{for}\ \ a,b>0; \tag{23}$$

$$aA-b\sqrt{A^2-G^2}\ge G\sqrt{a^2-b^2}\ \ \textit{for}\ \ a,b>0; \tag{24}$$

$$a\sqrt[4]{1-\frac{G^2}{A^2}}+b\sqrt{\frac{G}{A}}\le\left(\sqrt[3]{a^4}+\sqrt[3]{b^4}\right)^{\frac{3}{4}}\quad \textit{for}\ \ a,b>0; \tag{25}$$

$$\begin{aligned}\frac{1}{a}+\frac{1}{b}\quad&\le\quad\frac{A}{\sqrt{a^2G^2+b^2\left(A^2-G^2\right)}}+\frac{A}{\sqrt{a^2\left(A^2-G^2\right)+b^2G^2}}\\ &\le\quad\frac{4}{\sqrt{2\left(a^2+b^2\right)}}\quad \textit{for}\ \ a,b>0;\end{aligned} \tag{26}$$

$$\left(1+\left(\frac{A}{\sqrt{A^2-G^2}}\right)^n\right)\left(1+\left(\frac{A}{G}\right)^n\right)\ge\left(1+\left(\sqrt{2}\right)^n\right)^2; \tag{27}$$

$$\left|\frac{2G^2-A^2}{A^2+G\sqrt{A^2-G^2}}\right|\le\frac{2}{\sqrt{3}}; \tag{28}$$

$$2^{\frac{\sqrt{A^2-G^2}}{A}}+2^{\frac{G}{A}}\ge\sqrt{2^{2-\sqrt{2}}}; \tag{29}$$

$$\frac{\sqrt{A^2-G^2}}{A}+\frac{\left|A^2-2G^2\right|}{A^2}\ge\frac{2}{\sqrt{3}}; \tag{30}$$

$$\begin{aligned}&\frac{\sqrt{A^2(x_1,y_1)-G^2(x_1,y_1)}}{A(x_1,y_1)}+\frac{\sqrt{A^2(x_2,y_2)-G^2(x_2,y_2)}}{A(x_2,y_2)}\\&+\sin^2\left(\frac{\sqrt{A^2(x_1,y_1)-G^2(x_1,y_1)}}{M(x_1,y_1)}+\frac{\sqrt{A^2(x_2,y_2)-G^2(x_2,y_2)}}{M(x_2,y_2)}\right)\\\le\ &\frac{9}{4};\end{aligned} \tag{31}$$

$$\begin{aligned}&\frac{G(x_1,y_1)}{A(x_1,y_1)}+\frac{G(x_2,y_2)}{A(x_2,y_2)}\\&+2\cos\left(\frac{\sqrt{A^2(x_1,y_1)-G^2(x_1,y_1)}}{M(x_1,y_1)}+\frac{\sqrt{A^2(x_2,y_2)-G^2(x_2,y_2)}}{M(x_2,y_2)}\right)\\\ge\ &-\frac{9}{4};\end{aligned} \tag{32}$$

$$\begin{aligned}&8\frac{G(x_1,y_1)}{A(x_1,y_1)}\cdot\frac{G(x_2,y_2)}{A(x_2,y_2)}\\&\times\cos\left(\frac{\sqrt{A^2(x_1,y_1)-G^2(x_1,y_1)}}{M(x_1,y_1)}+\frac{\sqrt{A^2(x_2,y_2)-G^2(x_2,y_2)}}{M(x_2,y_2)}\right)+1\ge 0;\end{aligned} \tag{33}$$

$$\begin{aligned}&\frac{G(x_1,y_1)}{A(x_1,y_1)}+\frac{G(x_2,y_2)}{A(x_2,y_2)}\\\le\ &3+\cos\sqrt{\frac{\left(A^2(x_1,y_1)-G^2(x_1,y_1)\right)\left(A^2(x_2,y_2)-G^2(x_2,y_2)\right)}{M(x_1,y_1)M(x_2,y_2)}}\end{aligned} \tag{34}$$

$$\pi+\sqrt{\pi\left(\pi-\frac{2\sqrt{A^2-G^2}}{M}\right)}\le\frac{4\sqrt{A^2-G^2}}{M}\cot\left(\frac{2\sqrt{A^2-G^2}}{\pi M}\right); \tag{35}$$

$$\begin{aligned}2+0.164\frac{\left(A^2-G^2\right)^2}{GM^3}&<\frac{M^2}{A^2}+\frac{M}{G}\\&<2+0.178\frac{\left(A^2-G^2\right)\sqrt{A^2-G^2}}{GM^2};\end{aligned} \tag{36}$$

$$\arcsin\left(\frac{\sqrt{A^2-G^2}}{3M}\right)+\arcsin\left(\frac{2}{3}\sqrt{\frac{A-G}{A}}\right)\le\frac{2\sqrt{A^2-G^2}}{3M}; \tag{37}$$

$$3M^4+A^4\ge\left(3M^2+G^2\right)A^2; \tag{38}$$

$$3M^3 + G^3 \geq \left(3M^2 + A^2\right) G; \tag{39}$$

$$A^3 + 6M^3 \geq \left(6M^2 + G^2\right) A; \tag{40}$$

$$\frac{2M}{A} + \frac{M}{G} > 3; \tag{41}$$

$$\frac{M^2}{A^2} + \frac{M}{G} > 2. \tag{42}$$

$$\frac{M^2}{A^2} + \frac{M}{G} > 2 + \frac{16\left(A^2 - G^2\right)^2}{\pi^4 M^3 G}; \tag{43}$$

$$\frac{M}{A} = \prod_{k=1}^{\infty} \cos\left(\frac{\sqrt{A^2 - G^2}}{2^k M}\right); \tag{44}$$

$$\frac{M}{A} = \prod_{k=1}^{\infty} \left(1 - \frac{A^2 - G^2}{k^2 \pi^2 M^2}\right); \tag{45}$$

$$\frac{1}{2A} \geq \frac{1}{\pi M} + \frac{\pi^2 M^2 - 4A^2 + 4G^2}{24\pi M^3}; \tag{46}$$

$$\frac{8\left(\pi M - \sqrt{A^2 - G^2}\right)}{\pi^2\left(\pi M - 2\sqrt{A^2 - G^2}\right)} \leq \frac{1}{G} \leq \frac{\pi^2 M}{\pi^2 M^2 - 4A^2 + 4G^2}; \tag{47}$$

$$2M\left(A^2 - 2G^2 + A\sqrt{A^2 - G^2}\right) \leq A^2\left(\pi M - \sqrt{A^2 - G^2}\right); \tag{48}$$

$$\frac{3}{\pi} - \frac{4\left(A^2 - G^2\right)}{\pi^3 M^2} \leq \frac{M}{A} \leq 1 - \frac{4(\pi - 2)\left(A^2 - G^2\right)}{\pi^3 M^2}; \tag{49}$$

$$M \geq \frac{1}{2}(A + G); \tag{50}$$

$$6M \geq 2A + 4G + \sqrt{A^2 - G^2}; \tag{51}$$

$$(2A + G) M \geq 2A(A + G); \tag{52}$$

$$2A^3 + 4AMG \geq 3A^2 M + 2AG^2 + MG^2 + AG\sqrt{A^2 - G^2}; \tag{53}$$

$$M^3 \geq A \cdot G^2; \tag{54}$$

$$(A + \alpha G) M \geq (\alpha + 1) GA, \quad \textit{for} \quad \alpha \geq 2 \quad \textit{and} \quad \sqrt{1 - \left(\frac{1 + \sqrt{1 + 4\alpha}}{2\alpha}\right)^2} \leq \frac{x - y}{x + y} < 1; \tag{55}$$

(56)

$$(A + \alpha G) M \leq (\alpha + 1) GA, \quad \textit{for} \quad \alpha \geq 2 \quad \textit{and} \quad \frac{x - y}{x + y} < \sqrt{1 - \left(\frac{1 + \sqrt{1 + 4\alpha}}{2\alpha}\right)^2};$$

$$GM^2 + G^2 \geq 2M(A + 2G); \tag{57}$$

$$\frac{1}{A}+\frac{1}{A+G}\leq\frac{2}{M}; \tag{58}$$

$$\frac{M^2}{A}+2M\geq\frac{4M^2}{A+G}+G; \tag{59}$$

$$\left(\frac{A}{\sqrt{A^2-G^2}}\sinh\left(\frac{\sqrt{A^2-G^2}}{A}\right)\right)^3<2-\frac{G}{A}<\left(\frac{A}{M}\right)^3; \tag{60}$$

$$\frac{1}{4}\left(1+\frac{G}{A}\right)^2\leq\left(\frac{M}{A}\right)^3\leq\left(\cos\sqrt{\frac{A^2-G^2}{5M^2}}\right)^5; \tag{61}$$

$$\left(\prod_{k=1}^{n}\sqrt{A^2\left(x_k,y_k\right)-G^2\left(x_k,y_k\right)}\right)^m+\left(\prod_{k=1}^{n}G\left(x_k,y_k\right)\right)^m \tag{62}$$
$$\leq\left(\prod_{k=1}^{n}A\left(x_k,y_k\right)\right)^m;\ \ \textit{where } m\in N^*$$

$$\frac{2\sqrt{A^2-G^2}}{A+\sqrt{A^2-G^2}}\leq\ln\left(1+\frac{\sqrt{A^2-G^2}}{A}\right)\leq\frac{\left(2A+\sqrt{A^2-G^2}\right)\sqrt{A^2-G^2}}{2A\left(A+\sqrt{A^2-G^2}\right)}; \tag{63}$$

$$\frac{2\sqrt{A^2-G^2}}{2M+\sqrt{A^2-G^2}}\leq\ln\left(1+\frac{\sqrt{A^2-G^2}}{M}\right)\leq\frac{\left(2M+\sqrt{A^2-G^2}\right)\sqrt{A^2-G^2}}{2\left(M+\sqrt{A^2-G^2}\right)M}; \tag{64}$$

$$\left(\frac{M}{A}\right)^2<\frac{\pi^2M^2-A^2+G^2}{\pi^2M^2+A^2-G^2} \tag{65}$$

Proof. The proofs for the above inequalities are the following:

(1) If $t\in(0,1)$, then the following inequality holds

$$t\ln\frac{1+t}{1-t}\geq 2\left(\arcsin t\right)^2.$$

Now, let $t=\frac{x-y}{x+y}$ and after some computation, we obtain the second inequality.

(2) We have

$$\frac{\sin x}{x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}\quad\text{for}\quad x\in\left(0,\frac{\pi}{2}\right).$$

Let $x=\arcsin t$, $t\in(0,1)$, then we have

$$\frac{t}{\arcsin t}\geq\frac{\pi^2-\arcsin^2 t}{\pi^2+\arcsin^2 t}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(3) We know that

$$\frac{3\cos x}{1+2\cos x}\leq\frac{\sin x}{x}\leq\frac{3}{4-\cos x}\quad\text{for}\quad x\in\left(0,\frac{\pi}{2}\right).$$

Since

$$\frac{\sin x}{x}\geq\frac{3\cos x}{1+2\cos x}\Longleftrightarrow\tan x\left(1+2\cos x\right)\geq 3x.$$

Let $f(x)=(1+2\cos x)\tan x-3x$. As $f'(x)>0\Longrightarrow f(x)\geq f(0)=0$.

$$\begin{aligned}\frac{\sin x}{x} &\leq \frac{3}{4-\cos x}\leq 1. \quad f(t)=3x-\sin x(4-\cos x),\\ f'(t) &= 2(\cos x-1)^2\geq 0\Longrightarrow f(x)\geq f(0)=0.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)\Longrightarrow$

$$\frac{3\sqrt{1-t^2}}{1+2\sqrt{1-t^2}}\leq\frac{t}{\arcsin t}\leq\frac{3}{4-\sqrt{1-t^2}}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(4) We have the inequality

$$\frac{x}{\sqrt{1+x^2}}\leq\sin x\leq x,\quad x\in\left[0,\frac{\pi}{2}\right]$$

for

$$\begin{aligned}\tan x &\geq x\geq\sin x\Longrightarrow\tan^2 x=\frac{\sin^2 x}{\cos^2 x}\geq x^2\\ &\Longleftrightarrow \sin^2 x\geq x^2\cos^2 x\Longleftrightarrow\sin^2 x\geq x^2\left(1-\sin^2 x\right)\\ &\Longleftrightarrow \left(1+x^2\right)\sin^2 x\geq x^2\Longleftrightarrow\sin x\geq\frac{x}{\sqrt{1+x^2}}.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)$ then

$$\frac{\arcsin t}{\sqrt{1+\arcsin^2 t}}\leq t\leq\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(5) We know that

$$\begin{aligned}\frac{2\cos x}{1+\cos x} &\leq \frac{\sin x}{x},\quad x\in\left(0,\frac{\pi}{2}\right)\\ &\Longleftrightarrow \tan x(1+\cos x)\geq 2x,\quad f(t)=(1+\cos x)\tan x-2x,\\ f'(t) &= \frac{(1-\cos x)\left(\cos x+\sin^2 x\right)}{2\cos^2 x}\geq 0\Longrightarrow f(t)\geq f(0)=0.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)$ then

$$\frac{2\sqrt{1-t^2}}{1+\sqrt{1-t^2}}\leq\frac{t}{\arcsin t}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(6) We have the inequality

$$\begin{aligned}\sqrt[3]{\cos x} &\leq \frac{\sin x}{x},\quad x\in\left(0,\frac{\pi}{2}\right),\\ \frac{\sin x}{x} &\geq 1-\frac{x^2}{6}\Longrightarrow\frac{\sin^3 x}{x^3}\geq 1-\frac{x^2}{2}+\frac{x^4}{12}-\frac{x^6}{716},\\ -\cos x &\geq -1+\frac{x^2}{2}-\frac{x^4}{12}\\ &\Longrightarrow \frac{\sin^3 x}{x^3}-\cos x\geq x^4\cdot\frac{9-x^2}{216},\\ x &\in \left(0,\frac{\pi}{2}\right),\ 0<x<\sqrt{3},\ x^2<3,\ \frac{\sin^3 x}{x^3}\geq\cos x.\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$ then

$$\sqrt[6]{1-t^2} \leq \frac{t}{\arcsin t}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(7) If

$$\begin{aligned}
x &> 2 \Longrightarrow 0 < \frac{\pi}{x} < \frac{\pi}{2} \Longrightarrow \tan\frac{\pi}{x} > \frac{\pi}{x} > \frac{3}{x}; \\
\cos^2 x &= \frac{1}{1+\tan^2 x}, \ \cos^2\frac{\pi}{x} < \frac{1}{1+\left(\frac{3}{x}\right)^2} \\
&\Longrightarrow \sin^2\frac{\pi}{x} = 1-\cos^2\frac{\pi}{x} > 1-\frac{x^2}{x^2+9} = \frac{9}{x^2+9}, \\
\frac{\pi}{x} &= y \Longrightarrow y \in \left(0, \frac{\pi}{2}\right) \\
\sin y &\geq \frac{3y}{\sqrt{\pi^2+9y^2}}.
\end{aligned}$$

Let $y = \arcsin t$, $t \in (0,1)$, then the following inequality holds

$$t \geq \frac{3\arcsin t}{\sqrt{\pi^2+9\arcsin^2 t}}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(8) We have

$$\begin{aligned}
\frac{2}{\sin 2x} &\geq 2 \geq 3x - x^2, \quad x \in \left(0, \frac{\pi}{2}\right) \\
x^3 - 3x + 3 &= (x-1)^2(x+2) \Longrightarrow \sin 2x \leq \frac{2}{3x-x^3} \\
&\Longrightarrow \sin x \cos x \leq \frac{1}{3x-x^3}.
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$ then we have

$$t\sqrt{1-t^2} \leq \frac{1}{3\arcsin t - \arcsin^3 t}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(9) We know that

$$\begin{aligned}
\sin(\cos x) &\leq \cos(\sin x), \quad x \in \left(0, \frac{\pi}{2}\right) \\
\sin x &\leq x \Longrightarrow \sin(\cos x) \leq \cos x \\
&\Longrightarrow \sin(\cos x) \leq \cos x \leq \cos(\sin x).
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$ then the following inequality holds

$$\sin\sqrt{1-t^2} \leq \cos t.$$

In this inequality let $t = \frac{x-y}{x+y}$

$$\Longrightarrow \sin\frac{G}{A} \leq \cos\frac{\sqrt{A^2-G^2}}{A}.$$

(10) We have the inequalities

$$\begin{aligned}\sin(\sin x) &\le \cos(\cos x), \quad x \in \left(0, \frac{\pi}{2}\right), \\ \sin(\cos x) &\le \cos(\sin x).\end{aligned}$$

Let

$$X \to \frac{\pi}{2} - x \Longrightarrow \sin(\sin x) \le \cos(\cos x).$$

Now, letting $x = \arcsin t$, $t \in (0,1)$ we have

$$\sin t \le \cos\sqrt{1-t^2}.$$

In this inequality let $t = \frac{x-y}{x+y}$ to obtain

$$\sin\frac{\sqrt{A^2 - G^2}}{A} \le \cos\frac{G}{A}.$$

(11) We know that

$$\begin{aligned}t &\in (0,1) \Longrightarrow \arcsin t \le \frac{t}{1-t^2}. \\ f(t) &= \frac{t}{1-t^2} - \arcsin t, \\ f'(t) &= \frac{1 + t^2 - \sqrt{1-t^2}\left(1-t^2\right)}{\left(1-t^2\right)^2} > 0 \\ 1 + t^2 &\ge 1 \ge \sqrt{1-t^2}\left(1-t^2\right) \\ &\Longrightarrow f(t) \ge f(0) = 0.\end{aligned}$$

Substituting x for t, we have

$$\begin{aligned}f'(x) &= \frac{1 + x^2 - \sqrt{1-x^2}\left(1-x^2\right)}{\left(1-x^2\right)^2} > 0, \\ 1 + x^2 &\ge 1 \ge \sqrt{1-x^2}\left(1-x^2\right) \Longrightarrow f(x) \ge f(0) = 0.\end{aligned}$$

In this inequality let $t = \frac{x-y}{x+y}$.

(12) Let $f(x) = x^2 - \arcsin\left(\sin^2 x\right)$, $x \in \left(0, \sqrt{\frac{\pi}{2}}\right)$. Then

$$\begin{aligned}f'(x) &= 2\sin x\left(\frac{x}{\sin x} - \frac{1}{\sqrt{1+\sin^2 x}}\right) > 0, \\ \frac{x}{\sin x} &\ge 1 \ge \frac{1}{\sqrt{1+\sin^2 x}} \Longrightarrow f(x) \ge f(0) = 0. \\ \sin\left(\sin^2 x\right) &\le x^2 \Longrightarrow \sin^2 x \le \sin x^2.\end{aligned}$$

Let $x = \arcsin t$, $t \in \left(0, \sin\sqrt{\frac{\pi}{2}}\right)$ then

$$t^2 \le \sin\left(\arcsin^2 t\right).$$

In this inequality let $t = \frac{x-y}{x+y}$.

(13) We have the inequality

$$\begin{aligned}\frac{\tan x}{x}-\frac{x}{\sin x} &= \frac{1}{x\sin x}\left(\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\cdot\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}-x^2\right)\\ &> \frac{1}{x\sin x}\left(\frac{2x}{2}\cdot\frac{2x}{2}-x^2\right)=0\\ \frac{\tan x}{x} &\geq \frac{x}{\sin x}\Longrightarrow\frac{\sin^2 x}{\cos^2 x}\geq x^2.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)\Longrightarrow$

$$\frac{t^2}{\sqrt{1-t^2}}\geq\arcsin^2 t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(14) Let

$$\begin{aligned}f(x) &= \sin x+\tan x-2x\Longrightarrow f'(x)=\cos x+\frac{1}{\cos^2 x}-2\\ &\geq \left(\cos x+\frac{1}{\cos x}\right)-2\geq 2-2=0\Longrightarrow f(x)\geq f(0)=0.\\ x &\in \left(0,\frac{\pi}{2}\right)\Longrightarrow\frac{\sin x+\tan x}{2}\geq x\\ &\Longrightarrow \sin x+\tan x\geq 2x.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)$, then the following inequality holds

$$t+\frac{t}{\sqrt{1-t^2}}\geq 2\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(15) Let

$$\begin{aligned}f(x) &= 2\sin x+\tan x-3x,\quad x\in\left(0,\frac{\pi}{2}\right)\\ f'(x) &= \frac{(\cos x-1)^2(2\cos x+1)}{\cos^2 x}\geq 0\Longrightarrow f(x)\geq f(0)=0\\ &\Longrightarrow 2\sin x+\tan x\geq 3x.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)$, then we have

$$2t+\frac{t}{\sqrt{1-t^2}}\geq 3\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(16) We know that

$$\begin{aligned}\frac{x}{2} &\leq \tan\frac{x}{2}\leq x,\quad x\in\left(0,\frac{\pi}{2}\right),\\ f(x) &= x-\tan\frac{x}{2},\ f'(x)=\frac{\cos x}{1+\cos x}\geq 0\Longrightarrow f(x)\geq f(0)=0.\end{aligned}$$

Let $x=\arcsin t$, $t\in(0,1)$, then

$$\frac{1}{2}\arcsin t\leq\frac{t}{1+\sqrt{1-t^2}}\leq\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(17) Let

$$\begin{aligned} f(x) &= \tan x - x - \frac{x^3}{3}, \quad f'(t) = \tan^2 x - x^2 \geq 0, \\ f(x) &\geq f(0) = 0, \\ g(x) &= \tan x - x - \frac{x^3}{3} - \frac{2x^5}{15}, \ g'(x) = \tan^2 x - x^2 - \frac{2x^4}{3} \geq 0, \\ \tan^2 x &\geq \left(x + \frac{x^3}{3}\right)^2 = x^2 + \frac{2x^4}{3} + \frac{x^6}{9} \geq x^2 + \frac{2x^4}{3}, \\ \tan x &\geq x + \frac{x^3}{3} + \frac{2x^5}{15} \geq \frac{3x}{3 - x^2} \Longrightarrow 1 + \frac{x^2}{3} + \frac{2x^4}{15} \geq \frac{3}{3 - x^2} \\ &\Longrightarrow 10x^4 - 5x^4 - 2x^6 \geq 0 \\ &\Longrightarrow x^2 \leq \frac{5}{2}, \ x < \frac{\pi}{2} \Longrightarrow x^2 < \frac{5}{2} \\ &\Longrightarrow \tan x - x \geq \frac{x^2 \tan x}{3} \geq \frac{x^3}{3}, \quad x \in \left(0, \frac{\pi}{2}\right). \end{aligned}$$

Let $x = \arcsin t$, $t \in (0, 1)$, then

$$\frac{t}{\sqrt{1 - t^2}} - \arcsin t \geq \frac{t}{3\sqrt{1 - t^2}} \arcsin t \geq \frac{1}{3} \arcsin^3 t.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(18) We know that

$$\begin{aligned} \ln^2(\cos x) &\leq x(\tan x - x), \quad x \in \left(0, \frac{\pi}{2}\right) \\ \tan x &\geq x, \quad |\ln(\cos x)| \leq \sqrt{x(\tan x - x)} \\ &\Longleftrightarrow \sqrt{x(\tan x - x)} + \ln(\cos x) \geq 0, \\ f(x) &= \sqrt{x(\tan x - x)} + \ln(\cos x), \\ f'(x) &= \frac{\left(\sqrt{x}\tan x - \sqrt{\tan x - x}\right)^2}{2\sqrt{x(\tan x - x)}} \geq 0, \\ f(x) &\geq f(0) = 0. \end{aligned}$$

Let $x = \arcsin t$, $t \in (0, 1)$, then

$$\ln^2\left(\sqrt{1 - t^2}\right) \leq \left(\frac{t}{\sqrt{1 - t^2}} - \arcsin t\right) \arcsin t.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(19) Let

$$\begin{aligned} f(x) &= x \tan \frac{x}{2} + \ln(\cos x), \quad x \in \left(0, \frac{\pi}{2}\right) \\ &\Longrightarrow f'(x) = \left(1 + \tan^2 \frac{x}{2}\right)\left(\frac{x}{2} \cdot \frac{\tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}\right) < 0, \\ \frac{\tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} &> \tan \frac{x}{2} > \frac{x}{2} \Longrightarrow f(x) \geq f(0) = 0 \\ &\Longrightarrow x \tan \frac{x}{2} + \ln(\cos x) \leq 0. \end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$, then

$$\Longrightarrow \frac{t}{1+\sqrt{1-t^2}} \arcsin t + \ln \sqrt{1-t^2} \le 0.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(20) Let

$$\begin{aligned}
\arctan \frac{nx}{1+(n+1)x^2} &\le \ln \sqrt{n+1}, \quad n \in \mathbb{N},\ x > 0. \\
f(x) &= \arctan \frac{nx}{1+(n+1)x^2}, \\
f'(x) &= \arctan \frac{n\left(1-(n+1)x^2\right)}{n^2x^2 + \left((n+1)x^2+1\right)^2}, \\
x_0 &= \frac{1}{\sqrt{n+1}} \Longrightarrow f'(x_0) = 0 \\
&\Longrightarrow \frac{nx}{1+(n+1)x^2} \le \arctan \frac{n}{2\sqrt{n+1}}, \\
g(t) &= \frac{1}{2}\ln(x+1) - \arctan \frac{n}{2\sqrt{n+1}}, \\
g'(t) &= \frac{\left(\sqrt{x+1}-1\right)^2}{2(x+1)(x+2)} \ge 0.
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$, then

$$\frac{n \arcsin t}{1+(n+1)\arcsin^2 t} \le \tan\left(\ln \sqrt{n+1}\right).$$

In this inequality let $t = \frac{x-y}{x+y}$.

(21) We know that

$$\begin{aligned}
\left|\tan^n x - \cot^n x\right| &\ge 2n\left|\cos 2x\right|, \quad x \in \left(0, \frac{\pi}{2}\right), \ n \in \mathbb{N}; \\
\left|\tan^n x - \cot^n x\right| &= \left|\tan x - \cot x\right| \left|\tan^{n-1} x + \tan^{n-2} x \cot x + \cdots + \cot^{n-1} x\right|.
\end{aligned}$$

However,

$$\begin{aligned}
&\left|\tan x - \cot x\right| = \left|\frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\right| = \frac{2\left|\cos 2x\right|}{\left|\sin 2x\right|} \ge 2\left|\cos 2x\right|, \\
&\left|\tan^{n-1} x + \tan^{n-2} x \cot x + \cdots + \cot^{n-1} x\right| \\
\ge\ & n \sqrt[n]{\tan^{n-1} x \cdot \tan^{n-2} x \cot x \cdot \cdots \cdot \cot^{n-1} x} \\
=\ & n.
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$, then

$$\left|\left(\frac{t}{\sqrt{1-t^2}}\right)^n - \left(\frac{\sqrt{1-t^2}}{t}\right)^n\right| \ge 2n\left|1-2t^2\right|.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(22) We have that

$$(\sin x + a\cos x)(\sin x + b\cos x) \le 1 + \left(\frac{a+b}{2}\right)^2,$$

$a, b, x \in \mathbb{R}$.

$$\begin{aligned}
\cos x &= 0 \Longrightarrow 1+\left(\frac{a+b}{2}\right)^2 \geq \sin^2 x,\\
\cos x &\neq 0 \Longrightarrow \cos^2 x\left(\tan x+a\right)\left(\tan x+b\right) \leq 1+\left(\frac{a+b}{2}\right)^2,\\
\tan x &= y \Longrightarrow y^2\left(\frac{a+b}{2}\right)^2+1+\left(\frac{a+b}{2}\right)^2-(a+b)\,y-ab \geq 0\\
&\Longleftrightarrow \left(\frac{a+b}{2}y-1\right)^2+\left(\frac{a-b}{2}\right)^2 \geq 0.
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$, then

$$\left(t+a\sqrt{1-t^2}\right)\left(t+b\sqrt{1-t^2}\right) \leq 1+\left(\frac{a+b}{2}\right)^2.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(23) For $x \in \left(0, \frac{\pi}{2}\right)$, $a, b > 0$

$$\begin{aligned}
\left(1+\frac{a}{\sin x}\right)\left(1+\frac{b}{\cos x}\right) &\geq \left(1+\sqrt{2ab}\right)^2,\\
1+\frac{a}{\sin x}+\frac{b}{\cos x}+\frac{ab}{\sin x\cos x} &\geq 1+2\sqrt{2ab}+2ab,\\
\frac{ab}{\sin x\cos x} &= \frac{2ab}{\sin^2 x} \geq 2ab;\\
\frac{a}{\sin x}+\frac{b}{\cos x} &\geq 2\sqrt{\frac{2ab}{\sin^2 x}} \geq 2\sqrt{2ab}.
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$, then

$$\left(1+\frac{a}{t}\right)\left(1+\frac{b}{\sqrt{1-t^2}}\right) \geq \left(1+\sqrt{2ab}\right)^2.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(24) Let $a > b > 0$, $x \in \left(0, \frac{\pi}{2}\right)$, then we have the inequality

$$\begin{aligned}
\frac{a-b\sin x}{\cos x} &\geq \sqrt{a^2-b^2},\\
a^2+b^2 &= \left(a\sin x+b\cos x\right)^2+\left(a\sin x-b\cos x\right)^2\\
&\Longrightarrow -\sqrt{a^2+b^2} \leq a\sin x+b\cos x \leq \sqrt{a^2+b^2}.
\end{aligned}$$

Let $x = \arcsin t$, $t \in (0,1)$, then

$$a-bt \geq \sqrt{a^2-b^2}\sqrt{1-t^2}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(25) Let $a, b > 0$, $x \in \left(0, \frac{\pi}{2}\right)$, then we have the inequality

$$
\begin{aligned}
& & a\sqrt{\sin x} + b\sqrt{\cos x} \le \left(\sqrt[3]{a^4} + \sqrt[3]{b^4}\right)^{\frac{3}{4}}, \\
\sqrt[3]{a} = \alpha, \quad \sqrt[3]{b} = \beta & \Longrightarrow & \alpha^3\sqrt{\sin x} + \beta^3\sqrt{\cos x} \le \left(\alpha^4 + \beta^4\right)^{\frac{3}{4}}, \\
\alpha^3\sqrt{\sin x} + \beta^3\sqrt{\cos x} & = & \alpha^2\left(\alpha\sqrt{\sin x}\right) + \beta^2\left(\beta\sqrt{\cos x}\right) \\
& \le & \sqrt{\left(\alpha^4 + \beta^4\right)\left(\alpha^2\sin x + \beta^2\cos x\right)} \le \sqrt[4]{\left(\alpha^4 + \beta^4\right)^3} \\
& \Longleftrightarrow & \alpha^2\sin x + \beta^2\cos x \le \sqrt{\alpha^4 + \beta^4}.
\end{aligned}
$$

However,

$$\alpha^2\sin x + \beta^2\cos x \le \sqrt{\left(\alpha^4 + \beta^4\right)\left(\sin^2 x + \cos^2 x\right)} = \sqrt{\alpha^4 + \beta^4}.$$

Let $x = \arcsin t$, $t \in (0, 1)$, then

$$a\sqrt{t} + b\sqrt[4]{1 - t^2} \le \left(\sqrt[3]{a^4} + \sqrt[3]{b^4}\right)^{\frac{3}{4}}.$$

(26) We know that

$$\sqrt{a^2\sin^2 x + b^2\cos^2 x} = \sqrt{\frac{a^2 + b^2}{2}}\sqrt{1 + ky},$$

where

$$k = \frac{a^2 - b^2}{a^2 + b^2} \in (-1, 1), \ y = \cos 2x \in [-1, 1].$$

Let

$$
\begin{aligned}
f(y) & = & \sqrt{\frac{2}{a^2 + b^2}}\left[(1 + ky)^{-\frac{1}{2}} + (1 - ky)^{-\frac{1}{2}}\right], \\
f'(y) & = & -\frac{k^2\left(k^2y^2 + 3\right)y}{\sqrt{\left(1 - k^2y^2\right)^3}\left(\sqrt{(1 - ky)^3} + \sqrt{(1 + ky)^3}\right)}, \\
f_{\max} & = & f(0) = \frac{2\sqrt{2}}{\sqrt{a^2 + b^2}}, \\
f_{\min} & = & f(-1) = f(1) = \frac{1}{a} + \frac{1}{b} \\
& \Longrightarrow & \frac{1}{a} + \frac{1}{b} \le \frac{1}{\sqrt{a^2\cos^2 x + b^2\sin^2 x}} + \frac{1}{\sqrt{a^2\sin^2 x + b^2\cos^2 x}} \\
& \le & \frac{4}{\sqrt{2\left(a^2 + b^2\right)}}.
\end{aligned}
$$

Alternatively, if $t = \sin x$, then

$$\frac{1}{a} + \frac{1}{b} \le \frac{1}{\sqrt{a^2\left(1 - t^2\right) + b^2t^2}} + \frac{1}{\sqrt{a^2t^2 + b^2\left(1 - t^2\right)}} \le \frac{4}{\sqrt{2\left(a^2 + b^2\right)}}.$$

In this let $t = \frac{x - y}{x + y}$.

(27) If $x \in \left(0, \frac{\pi}{2}\right)$, then we have

$$\begin{aligned}\left(1+\frac{1}{\sin^n x}\right)\left(1+\frac{1}{\cos^n x}\right) &\geq \left(1+\left(\sqrt{2}\right)^n\right)^2,\\ \left(1+\frac{1}{\sin^n x}\right)\left(1+\frac{1}{\cos^n x}\right) &= 1+\frac{1}{\sin^n x}+\frac{1}{\cos^n x}+\frac{1}{\sin^n x\cos^n x}\\ &\geq 1+2\cdot 2^{\frac{n}{2}}+2^n=\left(1+\left(\sqrt{2}\right)^n\right)^2\end{aligned}$$

because

$$\frac{1}{\sin^n x\cos^n x}=\frac{2^n}{\sin^n 2x}\geq 2^n$$

and

$$\frac{1}{\sin^n x}+\frac{1}{\cos^n x}\geq 2\sqrt{\frac{1}{\sin^n x\cos^n x}}=2\cdot 2^{\frac{n}{2}}.$$

Let $t=\sin x$, then we obtain

$$\left(1+\frac{1}{t^n}\right)\left(1+\left(\frac{1}{\sqrt{1-t^2}}\right)^n\right)\geq\left(1+\left(\sqrt{2}\right)^n\right)^2.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(28) The following inequality holds for all $x \in \mathbb{R}$:

$$\begin{aligned}\left|\frac{\cos^2 x-\sin^2 x}{1+\sin x\cos x}\right| &\leq \frac{2}{\sqrt{3}}\Longleftrightarrow\left|\frac{\cos 2x}{2+\sin^2 x}\right|\leq\frac{1}{\sqrt{3}}\Longleftrightarrow\sqrt{3}\left|\cos 2x\right|\leq 2+2\sin 2x\\ &\Longleftrightarrow \left(2\sin 2x+1\right)^2\geq 0.\end{aligned}$$

Let $t=\sin x$, then we have

$$\left|\frac{1-2t^2}{1+t\sqrt{1-t^2}}\right|\leq\frac{2}{\sqrt{3}}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(29) We have

$$2^{\sin x}+2^{\cos x}\geq 2\cdot 2^{\frac{\sin x+\cos x}{2}}=2\cdot 2^{\frac{\sqrt{2}}{2}\sin\left(x+\frac{\pi}{4}\right)}\geq 2\cdot 2^{-\frac{\sqrt{2}}{2}}=\sqrt{2^{1-\sqrt{2}}}$$

because

$$\sin\left(x+\frac{\pi}{4}\right)\geq -1.$$

If we let $\sin x=t$, then we get

$$2^t+2^{\sqrt{1-t^2}}\geq\sqrt{2^{2-\sqrt{2}}}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(30) The following inequality holds for all $x \in \mathbb{R}$,

$$|\cos x| + |\cos 2x| \geq \frac{2}{\sqrt{2}};$$

$$\begin{aligned}
t &= |\cos x| \in [0,1] \Rightarrow \left|2t^2 - 1\right| + t \geq \frac{\sqrt{2}}{2};\\
t &\in \left[0, \frac{\sqrt{2}}{2}\right] \Rightarrow 1 - 2t^2 + t \geq \frac{\sqrt{2}}{2} \iff \left(t - \frac{\sqrt{2}}{2}\right)\left(2\left(t + \frac{\sqrt{2}}{2}\right) - 1\right) \leq 0,\\
t &\in \left(\frac{\sqrt{2}}{2}, 1\right] \Rightarrow 2t^2 - 1 + t - \frac{\sqrt{2}}{2} = \left(t - \frac{\sqrt{2}}{2}\right)\left(2\left(t + \frac{\sqrt{2}}{2}\right) + 1\right) \geq 0.
\end{aligned}$$

Thus, we have that

$$|t| + \left|2t^2 - 1\right| \geq \frac{\sqrt{2}}{2}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(31) The following inequality is well known.

$$\begin{aligned}
&\sin^2 x + \sin^2 y + \sin^2(x+y) \leq \frac{9}{4}\\
\iff &\frac{1-\cos 2x}{2} + \frac{1-\cos 2y}{2} + \frac{1-\cos 2(x+y)}{2} \leq \frac{9}{4}\\
\iff &\left(\cos x + \frac{1}{2}\cos(2y+x)\right)^2 + \frac{1}{4}\sin^2(2y+x) \geq 0.
\end{aligned}$$

Now, letting $x = \arcsin t_1$ and $y = \arcsin t_2$, we get

$$t_1 + t_2 + \sin^2(\arcsin t_1 + \arcsin t_2) \leq \frac{9}{4}.$$

In this inequality let $t_1 = \frac{x_1-y_1}{x_1+y_1}$ and $t_2 = \frac{x_2-y_2}{x_2+y_2}$.

(32) We have the following inequality

$$\begin{aligned}
&\cos x + \cos y + 2\cos(x+y) \geq -\frac{9}{4}\\
\iff &\left(4\cos\frac{x+y}{2} + \cos\frac{x-y}{2}\right)^2 + \sin^2\frac{x-y}{2} \geq 0.
\end{aligned}$$

Now, letting $x = \arcsin t_1$ and $y = \arcsin t_2$, we obtain

$$\sqrt{1-t_1^2} + \sqrt{1-t_2^2} + 2\cos(\arcsin t_1 + \arcsin t_2) \geq -\frac{9}{4}.$$

In this inequality let $t_1 = \frac{x_1-y_1}{x_1+y_1}$ and $t_2 = \frac{x_2-y_2}{x_2+y_2}$.

(33) We have the following inequality

$$\begin{aligned}
&(\cos 2x + \cos 2y) + \cos 2(x+y) \geq -\frac{3}{2}\\
\iff &8\cos x\cos y\cos(x+y) + 1 \geq 0.
\end{aligned}$$

Putting $x = \arcsin t_1$ and $y = \arcsin t_2$, we get

$$8\sqrt{1-t_1^2}\sqrt{1-t_2^2}\cos(\arcsin t_1 + \arcsin t_2) + 1 \geq 0.$$

In this inequality let $t_1 = \frac{x_1-y_1}{x_1+y_1}$ and $t_2 = \frac{x_2-y_2}{x_2+y_2}$.

(34) For all $x, y \in \mathbb{R}$ we have the inequality
$$\cos\left(x^2\right) + \cos\left(y^2\right) \le 3 + \cos(xy).$$
Since $\cos\left(x^2\right) \le 1$, $\cos\left(y^2\right) \le 1$, and $-\cos(xy) \le 1$, then
$$\cos\left(x^2\right) + \cos\left(y^2\right) - \cos(xy) \le 3.$$
If in the above inequality we put $x = \sqrt{a}$ and $y = \sqrt{b}$, then we get
$$\cos a + \cos b \le 3 + \cos\sqrt{ab}.$$
In additon, if we let $a = \arcsin t_1$ and $b = \arcsin t_2$, then we obtain
$$\sqrt{1 - t_1^2} + \sqrt{1 - t_2^2} \le 3 + \cos\left(\sqrt{\arcsin t_1 \cdot \arcsin t_2}\right).$$
In this inequality let $t_1 = \frac{x_1 - y_1}{x_1 + y_1}$ and $t_2 = \frac{x_2 - y_2}{x_2 + y_2}$.

(35) The following inequality is well known:
$$\pi + \sqrt{\pi(\pi - 2x)} \le 4x \cot\frac{2x}{\pi}, \quad x \in \left(0, \frac{\pi}{2}\right).$$
Substituting $x = \arcsin t$, we get
$$\pi + \sqrt{\pi(\pi - 2\arcsin t)} \le 4 \arcsin t \cot\left(\frac{2}{\pi}\arcsin t\right).$$
In this inequality let $t = \frac{x-y}{x+y}$.

(36) We have the inequality
$$\begin{aligned} 2 + 0.164\frac{x^3 \sin x}{\cos x} &< \left(\frac{\sin x}{x}\right)^2 + \frac{\sin x}{x\cos x} \\ &< 2 + 0.178\frac{x^2 \sin x}{\cos x}. \end{aligned}$$
Letting $x = \arcsin t$, we have
$$\begin{aligned} 2 + 0.164\frac{t}{\sqrt{1-t^2}}\arcsin^3 t &< \left(\frac{t}{\arcsin t}\right)^2 + \frac{t}{\sqrt{1-t^2}\arcsin t} \\ &< 2 + 0.178\frac{t}{\sqrt{1-t^2}}\arcsin^2 t. \end{aligned}$$
In this inequality let $t = \frac{x-y}{x+y}$.

(37) G. Baloglou in A.M.M. E. 3326 proved that if $0 < x < \pi$, then
$$\arcsin\frac{x}{6} + \arcsin\left(\frac{2}{3}\sin\frac{x}{4}\right) \le \frac{x}{3}.$$
Let $x \to 2y$, then $0 < y < \frac{\pi}{2}$ implies that
$$\arcsin\frac{y}{3} + \arcsin\left(\frac{2}{3}\sin\frac{y}{2}\right) \le \frac{2y}{3}.$$
Letting $y = \arcsin t$, we get
$$\arcsin\left(\frac{1}{3}\arcsin t\right) + \arcsin\left(\frac{2}{3}\sqrt{\frac{1-\sqrt{1-t^2}}{2}}\right) \le \frac{2}{3}\arcsin t.$$
In this inequality let $t = \frac{x-y}{x+y}$.

(38) It is well known that $x \in \left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{\sin x}{x}\right)^2 \geq 1 - \frac{x^2}{3}.$$

Putting $x = \arcsin t$, we get

$$\left(\frac{t}{\arcsin t}\right)^2 \geq 1 - \frac{1}{3}\left(\arcsin t\right)^2.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(39) We have the inequality

$$\frac{\tan x}{x} \geq 1 + \frac{x^2}{3}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Letting $x = \arcsin t$, we get

$$\left(\frac{t}{\sqrt{1-t^2}\arcsin t}\right)^2 \geq 1 + \frac{1}{3}\left(\arcsin t\right)^2.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(40) We have the inequality

$$\frac{\sin x}{x} \geq 1 - \frac{x^2}{6}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Put $x = \arcsin t$, then

$$\frac{t}{\arcsin t} \geq 1 - \frac{1}{6}\left(\arcsin t\right)^2.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(41) We have the inequality

$$\frac{\sin x}{x} + \frac{\sin x}{2x\cos x} \geq \frac{3}{2}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Substituting $x = \arcsin t$, we get

$$\frac{t}{\arcsin t} + \frac{t}{2\sqrt{1-t^2}\arcsin t} \geq \frac{3}{2}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(42) We have

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad x \in \left(0, \frac{\pi}{2}\right).$$

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Putting $x = \arcsin t$, we have

$$\left(\frac{t}{\arcsin t}\right)^2 + \frac{t}{\sqrt{1-t^2}\arcsin t} > 2.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(43) We know that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\sin x}{x\cos x} \geq 2 + \frac{16x^3\sin x}{\pi^4\cos x}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Let $x = \arcsin t$, then

$$\left(\frac{t}{\arcsin t}\right)^2 + \frac{t}{\sqrt{1-t^2}\arcsin t} > 2 + \frac{16t}{\pi^4\sqrt{1-t^2}}\left(\arcsin t\right)^3.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(44) The following identity is well known

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos\frac{x}{2^k}.$$

Putting $x = \arcsin t$, we have

$$\frac{t}{\arcsin t} = \prod_{k=1}^{\infty} \cos\left(\frac{\arcsin t}{2^k}\right).$$

In this inequality let $t = \frac{x-y}{x+y}$.

(45) It is known that

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right).$$

Substituting $x = \arcsin t$, we obtain

$$\frac{t}{\arcsin t} = \prod_{k=1}^{\infty}\left(1 - \frac{\left(\arcsin t\right)^2}{k^2\pi^2}\right).$$

In this inequality let $t = \frac{x-y}{x+y}$.

(46) The following inequality holds for $x \in \left(0, \frac{\pi}{2}\right)$,

$$\sin x \geq \frac{2x}{\pi} + \frac{1}{12\pi} \times \left(\pi^2 - 4x^2\right), \quad x \in \left(0, \frac{\pi}{2}\right).$$

Putting $x = \arcsin t$, we get

$$t \geq \frac{2}{\pi}\arcsin t + \frac{1}{12\pi}\arcsin t\left(\pi^2 - 4\left(\arcsin t\right)^2\right).$$

In this inequality let $t = \frac{x-y}{x+y}$.

(47) The following inequality holds for $x \in \left(0, \frac{\pi}{2}\right)$,

$$\frac{8\left(\pi - x\right)x}{\pi^2\left(\pi - 2x\right)} \leq \frac{\sin x}{\cos x} \leq \frac{\pi^2 x}{\pi^2 - 4x^2}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Substituting $x = \arcsin t$, we have

$$\frac{8\left(\pi - \arcsin t\right)\arcsin t}{\pi^2\left(\pi - 2\arcsin t\right)} \leq \frac{t}{\sqrt{1-t^2}} \leq \frac{\pi^2\arcsin t}{\pi^2 - 4\left(\arcsin t\right)^2}.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(48) The following inequality holds

$$2\sin^2 x + \sin x - 1 \leq \frac{\pi}{2} - x, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Putting $x = \arcsin t$, we get

$$2t^2 + t - 1 \leq \frac{\pi}{2} - \arcsin t.$$

In this inequality let $t = \frac{x-y}{x+y}$.

(49) The following inequality holds

$$\frac{3}{\pi}-\frac{4x^2}{\pi^3}<\frac{\sin x}{x}<1-\frac{4(\pi-2)x^2}{\pi^3},\quad x\in\left(0,\frac{\pi}{2}\right).$$

See [5]. Letting $x=\arcsin t$, we obtain

$$\frac{3}{\pi}-\frac{4(\arcsin t)^2}{\pi^3}\le\frac{t}{\sin t}\le 1-\frac{4(\pi-2)(\arcsin t)^2}{\pi^3}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(50) We know that

$$\frac{x}{\sin x}\le\frac{2}{1+\cos x},\quad x\in\left[0,\frac{\pi}{2}\right].$$

See [5]. Substituting $x=\arcsin t$, we have

$$2t\ge\arcsin t+\sqrt{1-t^2}\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(51) We have

$$6\frac{\sin x}{x}\ge 2+4\cos x+\sin x,\quad x\in\left(0,\frac{\pi}{2}\right).$$

Let $x=\arcsin t$, then

$$6t\ge\left(2+4\sqrt{1-t^2}+t\right)\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(52) It is common knowledge that

$$\sin^2 x\le 2x\sin x+\cos x-1,\quad x\in\left(0,\frac{\pi}{2}\right).$$

Putting $x=\arcsin t$, we obtain

$$t^2\le 2t\arcsin t+\sqrt{1-t^2}-1.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(53) We have the inequality

$$\sin^2 x+2x\sin x\ge 4-4\cos x+x\cos x,\quad x\in\left(0,\frac{\pi}{2}\right).$$

Substituting $x=\arcsin t$, we get

$$t^2+2t\arcsin t\ge 4-4\sqrt{1-t^2}+\sqrt{1-t^2}\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(54) Let

$$\begin{aligned}
f(t) &= \frac{\sin}{\sqrt[3]{\cos x}}-x,\quad \frac{1+\cos^2 x+\cos^2 x}{3}\ge(\cos x)^{\frac{1}{3}};\\
f'(t) &= \frac{1+2\cos^2 x}{3(\cos x)^{\frac{4}{3}}}-1\ge 0\\
&\Longrightarrow\quad f(x)\ge f(0)\Longrightarrow\left(\frac{\sin x}{x}\right)^3\ge\cos x,\quad x\in\left(0,\frac{\pi}{2}\right).
\end{aligned}$$

Let $x=\arcsin t$, then

$$\left(\frac{t}{\arcsin t}\right)^3 \geq \sqrt{1-t^2}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(55) If $t\in\left(\sqrt{1-\left(\frac{1+\sqrt{1+4\alpha}}{2\alpha}\right)^2},1\right)$, then the following inequality holds [6]:

$$\left(1+\alpha\sqrt{1-t^2}\right)\frac{t}{\sqrt{1-t^2}} \geq (\alpha+1)\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(56) If $t\in\left(0,\sqrt{1-\left(\frac{1+\sqrt{1+4\alpha}}{2\alpha}\right)^2}\right)$, then the following inequality holds [6]:

$$\left(1+\alpha\sqrt{1-t^2}\right)\frac{t}{\sqrt{1-t^2}} \leq (\alpha+1)\arcsin t.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(57) We have

$$\frac{\sin x}{x} \geq \frac{1+2\cos x}{3}+\frac{x\sin x}{6}, \quad x\in\left[0,\frac{\pi}{2}\right].$$

See [5]. Put $x=\arcsin t$, then we have

$$\frac{t}{\arcsin t} \geq \frac{1+2\sqrt{1-t^2}}{3}+\frac{t\arcsin t}{6}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(58) The following inequality holds

$$\left(\frac{\sin x}{x}\right)^2 \leq 2\left(\frac{\sin x}{x}+\frac{\cos x-1}{x^2}\right), \ \ x\in\left(0,\frac{\pi}{2}\right).$$

See [5]. Letting $x=\arcsin t$, we obtain

$$\left(\frac{t}{\arcsin t}\right)^2 \leq 2\left(\frac{t}{\arcsin t}+\frac{\sqrt{1-t^2}-1}{(\arcsin t)^2}\right).$$

In this inequality let $t=\frac{x-y}{x+y}$.

(59) We have the inequality

$$\left(\frac{\sin x}{x}\right)^2+2\frac{\sin x}{x} \geq 4\left(\frac{1-\cos x}{x^2}\right)+\cos x, \ \ x\in\left(0,\frac{\pi}{2}\right).$$

See [5]. Putting $x=\arcsin t$, we get

$$\left(\frac{t}{\arcsin t}\right)^2+2\left(\frac{t}{\arcsin t}\right) \geq 4\frac{1-\sqrt{1-t^2}}{(\arcsin t)^2}+\sqrt{1-t^2}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(60) The following inequality holds

$$\left(\frac{\sinh x}{x}\right)^3 < 2-\sqrt{1-x^2} \le \left(\frac{\arcsin x}{x}\right)^3, \ x\in(0,1)\,.$$

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In this inequality let $x=\frac{x-y}{x+y}$.

(61) We have

$$\left(\cos\frac{x}{2}\right)^4 \le \left(\frac{\sinh x}{x}\right)^3 \le \left(\cos\frac{x}{\sqrt{5}}\right)^5, \ x\in(0,\pi)\,.$$

Put $x=\arcsin t$, then

$$\left(\frac{\sqrt{1+t}+\sqrt{1-t}}{2}\right)^4 \le \left(\frac{t}{\arcsin t}\right)^3 \le \left(\cos\left(\frac{\arcsin t}{\sqrt{5}}\right)\right)^5.$$

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In this inequality let $t=\frac{x-y}{x+y}$.

(62) The following inequality holds

$$\prod_{k=1}^n \sin x_k^m + \prod_{k=1}^n \cos x_k^m \le \prod_{k=1}^n \sin x_k + \prod_{k=1}^n \cos x_k \le 1, \ x_k\in\left(0,\frac{\pi}{2}\right),$$

by induciton.
Let $x_k=\arcsin t_k$, then

$$\left(\prod_{k=1}^n t_k\right)^m + \left(\prod_{k=1}^n \sqrt{1-t_k^2}\right)^m \le 1.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(63) If $t\in(0,1)$, then the following inequality holds

$$\frac{2t}{t+2} \le \ln(t+1) \le \frac{t\,(t+2)}{2\,(t+1)}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(64) If $t\in(0,1)$, then the following inequality holds

$$\frac{2\arcsin t}{2+\arcsin t} \le \ln(1+\arcsin t) \le \frac{(2+\arcsin t)\arcsin t}{2\,(1+\arcsin t)}.$$

In this inequality let $t=\frac{x-y}{x+y}$.

(65) If $t\in(0,1)$, then the following inequality holds

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\pi^2-x^2}{\pi^2+x^2}, \quad x\in\left(0,\frac{\pi}{2}\right).$$

See Problem 2090, Crux (1995, 307).
Let $x=\arcsin t$, $t\in(0,1)$

$$\left(\frac{t}{\arcsin t}\right)^2 < \frac{\pi^2-\arcsin^2 t}{\pi^2+\arcsin^2 t}.$$

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