PERTURBED RULES IN NUMERICAL INTEGRATION FROM PRODUCT BRANCHED PEANO KERNELS

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Abstract. Perturbed rules are obtained by using a variety of inequalities to obtain bounds for the Chebychev functional. In particular, Grüss, premature Grüss seminorms are used to obtain bounds for perturbed quadrature rules involving the boundary points and an interior point. Generalised Simpson type rules are shown to be recaptured as special instances of the current development.

1. Introduction

Recently, Cerone [1] obtained the following identity involving n-time differentiable functions with evaluation at an interior point and at the end points.

Lemma 1. *Let r_k, s_k, u_k, $v_k \in \mathcal{A}$ for $k \in \mathbb{N}$ be sequences of polynomials which are such that $w_k \in \mathcal{A}$ if*

$$w_k'(t) = w_{k-1}(t), \;\; w_0(t) = 1, \;\; t \in \mathbb{R}. \tag{1.1}$$

Further, define $K_n(x,t)$, $p_n(t)$ and $q_n(t)$ by

$$K_n(x,t) = \begin{cases} p_n(t) = r_{n-m}(t)\, s_m(t), & t \in [a,x] \\ \\ q_n(t) = u_{n-m}(t)\, v_m(t), & t \in (x,b]. \end{cases} \tag{1.2}$$

Then for $f : [a,b] \to \mathbb{R}$ and $f^{(n-1)}$ absolutely continuous on $[a,b]$, the identity

$$\begin{aligned} &(-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\, dt \\ = & \binom{n}{m} \int_a^b f(t)\, dt + \sum_{k=0}^{n-1} (-1)^k \left\{ \left[p_n^{(k)}(x) - q_n^{(k)}(x) \right] f^{(n-1-k)}(x) \right. \\ & \left. + q_n^{(k)}(b)\, f^{(n-1-k)}(b) - p_n^{(k)}(a)\, f^{(n-1-k)}(a) \right\} \end{aligned} \tag{1.3}$$

holds, where

$$\begin{cases} p_n^{(k)}(\cdot) = \sum_{j=L}^{U} \binom{k}{j} r_{n-m-j}(\cdot)\, s_{m-k+j}(\cdot), \\ \\ q_n^{(k)}(\cdot) = \sum_{j=L}^{U} \binom{k}{j} u_{n-m-j}(\cdot)\, v_{m-k+j}(\cdot) \end{cases} \tag{1.4}$$

Date: November 15, 2000.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 65Xxx.

Key words and phrases. Three point identities and inequalities, Ostrowski and Grüss type inequalities, Newton-Cotes Quadrature, Chebychev functional.

with

$$U=\min\{k,n-m\},\ L=\max\{0,k-m\}. \tag{1.5}$$

In an earlier paper Pearce et al. [13] assumed that $p_n(\cdot)$ and $q_n(\cdot)$ in (1.2) satisfy (1.1) which Cerone [1] termed as being Appell-like polynomials. The above lemma stipulates that $p_n(\cdot)$ and $q_n(\cdot)$ are comprised of the products of functions satisfying (1.1). The following theorems were derived in Cerone [1], giving bounds for various three point rules. For other articles related to three point rules see [4] – [6] and [10], while [7] gives trapezoidal rules all of which, are particular instances of Cerone [1].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. The following inequalities hold for $f^{(n)}\in L_p[a,b]$, $p\geq 1$ and for all $x\in[a,b]$,*

$$\begin{aligned} |\tau_n(x)| &:= \left|\binom{n}{m}\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}(-1)^{k+1}\left\{\left[p_n^{(k)}(x)\right.\right.\right.\\ &\quad\left.\left.\left.-q_n^{(k)}(x)\right]f^{(n-1-k)}(x)+q_n^{(k)}(b)\,f^{(n-1-k)}(b)-p_n^{(k)}(a)\,f^{(n-1-k)}(a)\right\}\right|\\ &\leq\begin{cases}\left\|f^{(n)}\right\|_\infty B_n(1,x) & \text{for}\quad f^{(n)}\in L_\infty[a,b];\\ \left\|f^{(n)}\right\|_p [B_n(q,x)]^{\frac{1}{q}} & \text{for}\quad f^{(n)}\in L_p[a,b],\\ & \text{with}\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left\|f^{(n)}\right\|_1\theta_n(x) & \text{for}\quad f^{(n)}\in L_1[a,b],\end{cases}\end{aligned} \tag{1.6}$$

where

$$B_n(q,x) = \int_a^x |p_n(t)|^q\,dt+\int_x^b |q_n(t)|^q\,dt \tag{1.7}$$

$$\theta_n(x) = \frac{M_n(a,x)+M_n(x,b)}{2}+\frac{|M_n(x,b)-M_n(a,x)|}{2}, \tag{1.8}$$

with $M_n(a,x)=\sup_{t\in[a,x]}|p_n(t)|,\ M_n(x,b)=\sup_{t\in(x,b]}|q_n(t)|,$
$p_n(t),\ q_n(t)$ *are defined by (1.2),*

$p_n^{(k)}(t),\ q_n^{(k)}(t)$ *are as given by (1.4),*
and the Lebesgue norms are defined by

$$\begin{cases}\left\|f^{(n)}\right\|_\infty := ess\sup_{t\in[a,b]}\left|f^{(n)}(t)\right|<\infty,\\ \left\|f^{(n)}\right\|_p := \left(\int_a^b\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}},\quad p\geq 1.\end{cases} \tag{1.9}$$

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. The following inequalities hold for $f^{(n)}\in L_p[a,b]$, $p\geq 1$ and for*

$\alpha, x, \beta \in [a,b]$ *with* $a \leq \alpha,\ x,\ \beta \leq b$,

$$\left|\tau_n^C(x)\right| := \left|\int_a^b f(t)\,dt - \frac{1}{\binom{n}{m}} \sum_{k=0}^{n-1} (-1)^{k+1} \left\{ \left[p_n^{(k)}(x) - q_n^{(k)}(x) \right] f^{(n-1-k)}(x) + q_n^{(k)}(b) f^{(n-1-k)}(b) - p_n^{(k)}(a) f^{(n-1-k)}(a) \right\} \right| \tag{1.10}$$

$$\leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{n!} \tilde{B}_n^C(1,x) & \text{for} \quad f^{(n)} \in L_\infty[a,b]; \\ \frac{\|f^{(n)}\|_p}{n!} \left[\tilde{B}_n^C(q,x)\right]^{\frac{1}{q}} & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & \text{with} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f^{(n)}\|_1}{n!} \tilde{\theta}_n^C(x) & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases}$$

where

$$\tilde{B}_n^C(q,x) = \begin{cases} (\alpha-a)^{nq+1} B((n-m)q+1, mq+1) \\ +(x-\alpha)^{nq+1} \Psi\left(mq, (n-m)q; \frac{\alpha-a}{x-\alpha}\right) \\ +(\beta-x)^{nq+1} \Psi\left(mq, (n-m)q; \frac{b-\beta}{\beta-x}\right) \\ +(b-\beta)^{nq+1} B(mq+1, (n-m)q+1), \quad a \leq \alpha < x < \beta \leq b \\ \\ (x-a)^{nq+1} \chi\left((n-m)q, mq, \frac{\alpha-a}{x-a}\right) \\ +(b-x)^{nq+1} \chi\left((n-m)q, mq, \frac{b-\beta}{\beta-x}\right), \qquad a \leq \beta < x < \alpha \leq b \end{cases} \tag{1.11}$$

$$\tilde{\theta}_n^C(x) = \begin{cases} \max\left\{A^{n-m}\left[\frac{A}{2}+\left|C-\frac{A}{2}\right|\right]^m, B^{n-m}\left[\frac{B}{2}+\left|D-\frac{B}{2}\right|\right]^m\right\}, \\ a \leq \alpha < x < \beta \leq b \\ \\ \max\left\{A^{n-m}(A-C)^m, B^{n-m}(B-D)^m\right\}, \quad a \leq \beta \leq x \leq \alpha \leq b \end{cases} \tag{1.12}$$

$$\begin{cases} p_n^{(k)}(t) = \sum_{j=L}^{U} \binom{k}{j} \frac{(t-a)^{n-m-j}}{(n-m-j)!} \cdot \frac{(t-\alpha)^{m-k+j}}{(m-k+j)!} \\ \text{and} \quad q_n^{(k)}(t) = \sum_{j=L}^{U} \binom{k}{j} \frac{(t-b)^{n-m-j}}{(n-m-j)!} \cdot \frac{(t-\beta)^{m-k+j}}{(m-k+j)!} \end{cases} \tag{1.13}$$

with U *and* L *as given by (1.5),*

$$\begin{cases} \Psi(j,k;X) = \int_0^1 u^j (X+u)^k\,du, \\ \chi(j,k;X) = \int_0^1 u^j (X-u)^k\,du, \\ B(j,k) = \chi(j-1,k-1;1) \text{ is the Euler beta function,} \end{cases} \tag{1.14}$$

and

$$A = x-a, \quad C = x-\alpha, \quad B = b-x, \quad D = \beta-x. \tag{1.15}$$

The proof of Theorem 2 involves taking $p_n(\cdot)$ and $q_n(\cdot)$ in (1.2) as

$$p_n^C(t) = \frac{(t-a)^{n-m}}{(n-m)!} \cdot \frac{(t-\alpha)^m}{m!} \quad \text{and} \quad q_n^C(t) = \frac{(t-b)^{n-m}}{(n-m)!} \cdot \frac{(t-\beta)^m}{m!}. \tag{1.16}$$

The above theorem was shown to be quite general, recapturing many previous results, including that of Fink [10].

Taking $\alpha = \lambda x + (1-\lambda)a$ and $\beta = \lambda x + (1-\lambda)b$ in Theorem 2, the following corollary was also obtained, giving the bounds in a simplified form involving the parameter λ.

Corollary 1. *Let the conditions of Theorem 2 hold. Then*

$$\begin{aligned} \left|\tau_n^{C^*}(x)\right| &:= \left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} (-1)^{k+1} d_n^{(k)}(x) f^{(n-1-k)}(x) - \right. \\ &\left. \sum_{k=n-m}^{n-1} (-1)^{k+1} \left[Q_n^{(k)}(b) f^{(n-1-k)}(b) - P_n^{(k)}(a) f^{(n-1-k)}(a) \right] \right| \\ &\leq \begin{cases} \frac{B_n^{C^*}(1,x)}{n!} \left\| f^{(n)} \right\|_\infty, & \text{for} \quad f^{(n)} \in L_\infty[a,b]; \\ \frac{\left[B_n^{C^*}(q,x)\right]^{\frac{1}{q}}}{n!} \left\| f^{(n)} \right\|_p, & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & \text{with} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\theta_n^{C^*}(x)}{n!} \left\| f^{(n)} \right\|_1, & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases} \end{aligned} \tag{1.17}$$

where

$$\begin{aligned} & B_n^{C^*}(q,x) \\ &= \left[A^{nq+1} + B^{nq+1}\right] \begin{cases} \lambda^{nq+1} B((n-m)q+1, mq+1) + \\ (1-\lambda)^{nq+1} \Psi\left(mq, (n-m)q, \frac{\lambda}{1-\lambda}\right), & 0 \leq \lambda < 1; \\ \chi((n-m)q, mq, \lambda), & \lambda \geq 1, \end{cases} \end{aligned} \tag{1.18}$$

$$\begin{aligned} & \theta_n^{C^*}(x) \\ &= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n \begin{cases} \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^m, & 0 \leq \lambda < 1; \\ \lambda^m, & \lambda \geq 1, \end{cases} \end{aligned} \tag{1.19}$$

with $\Psi(j,k;X)$, $\chi(j,k;X)$ *and* $B(j,k)$ *as given by (1.14) and* $A = x-a$, $B = b-x$. *Further,*

$$d_n^{(k)}(x) = \left[A^{n-k} - (-B)^{n-k}\right] \frac{1}{\binom{n}{m}} \sum_{j=L}^{U} \frac{\binom{k}{j}(1-\lambda)^{m-k+j}}{(n-m-j)!(m-k+j)!},$$

$$P_n^{(k)}(a) = A^{n-k}(-\lambda)^{n-k} \frac{\binom{k}{n-m}}{\binom{n}{m}}, \quad k \geq n-m \tag{1.20}$$

$$\text{and} \quad Q_n^{(k)}(b) = B^{n-k}\lambda^{n-k} \frac{\binom{k}{n-m}}{\binom{n}{m}}, \quad k \geq n-m$$

with, from (1.5), $U = \min\{k, n-m\}$ *and* $L = \max\{0, k-m\}$.

Finally, Cerone [1] obtained the following corollary by taking $m = 1$ in (1.16), and allowing the dependence on x of α and β to be shown explicitly and subscripted to denote their dependence on n. Further, a superscript of S was used to depict the relationship with Simpson's rule in the branches of the Peano kernel, and so

$$p_n^S(t) = \frac{(t-a)^{n-1}}{(n-1)!}(t-\alpha_n(x)), \quad q_n^S(t) = \frac{(t-b)^{n-1}}{(n-1)!}(t-\beta_n(x)), \tag{1.21}$$

with

$$\alpha_n(x) = \lambda_n x + (1-\lambda_n)a, \quad \beta_n(x) = \lambda_n x + (1-\lambda_n)b, \quad \lambda_n = \frac{n}{3}. \tag{1.22}$$

Corollary 2. *Let the conditions of Corollary 1 hold, then*

$$\begin{aligned} \left|\tau_n^S(x)\right| \quad : \quad &= \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}(-1)^{k+1} d_n^{(k)}(x) f^{(n-1-k)}(x) - \right. \\ &\quad \left. \frac{(-1)^n \lambda_n}{n}\left[B \cdot f(b) + A \cdot f(a)\right]\right| \\ &\leq \begin{cases} \frac{B_n^S(1,x)}{n!}\left\|f^{(n)}\right\|_\infty, & \text{for} \quad f^{(n)} \in L_\infty[a,b]; \\[2ex] \frac{\left[B_n^S(q,x)\right]^{\frac{1}{q}}}{n!}\left\|f^{(n)}\right\|_p, & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & \text{with} \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\theta_n^S(x)}{n!}\left\|f^{(n)}\right\|_1, & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases} \end{aligned} \tag{1.23}$$

where

$$\begin{aligned} &B_n^S(q,x) \\ &= \left[A^{nq+1} + B^{nq+1}\right] \begin{cases} \lambda_n^{nq+1} B((n-1)q+1, q+1) + \\ (1-\lambda_n)^{nq+1}\,\Psi\left(q, (n-1)q, \frac{\lambda_n}{1-\lambda_n}\right), & 0 \leq \lambda_n < 1; \\[2ex] \chi((n-1)q, q, \lambda_n), & \lambda_n \geq 1, \end{cases} \end{aligned} \tag{1.24}$$

$$\theta_n^S(x) = \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n \begin{cases} \frac{1}{2} + \left|\lambda_n - \frac{1}{2}\right|, & 0 \le \lambda_n < 1; \\ \lambda_n, & \lambda_n \ge 1, \end{cases} \tag{1.25}$$

$$d_n^{(k)}(x) = \left[A^{n-k} - (-B)^{n-k}\right] \frac{1}{n} \sum_{j=\max\{0,k-1\}}^{\min\{k,n-1\}} \frac{\binom{k}{j}(1-\lambda_n)}{(n-1-j)!\,(1-k+j)!}, \tag{1.26}$$

with $\Psi(j,k;X)$, $\chi(j,k;X)$ *and* $B(j,k)$ *as given by (1.14) and* $A = x-a$, $B = b-x$, $\lambda_n = \frac{x}{3}$.

Proof. Taking $p_n^S(t)$, $q_n^S(t)$, $\alpha_n(x)$, $\beta_n(x)$ and $\lambda_n = \frac{n}{3}$ produces the above results after some simplification. ∎

If $n = 1,2,3$ and 4, then we obtain the Simpson type results of Dragomir et al. [9] and Pečarić Varošanec [14], provided that x is taken at the midpoint, that is, at $x = \frac{a+b}{2}$.

It is the express aim of this paper to present perturbed rules for $\tau_n(x)$, $\tau_n^C(x)$, $\tau_n^{C^*}(x)$ and $\tau_n^S(x)$ of Theorems 1 and 2 and Corollaries 1 and 2 respectively.

Perturbed rules using Grüss type inequalities involving the Chebychev functional are investigated in Section 2. The idea of bounds in terms of $\Delta-$seminorms is briefly canvassed, although this is left as future work. In Section 3 the implementation of the results as composite quadrature rules is demonstrated. This allows the determination of the partition required to achieve a given error tolerance.

2. Perturbed Results

Perturbed versions of the results of the previous sections may be obtained by using Grüss type results involving the Chebychev functional

$$T(f,g) = \mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g) \tag{2.1}$$

with

$$\mathfrak{M}(f) = \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{2.2}$$

For $f,g : [a,b] \to \mathbb{R}$ and integrable on $[a,b]$, as is their product, then

$$\begin{aligned} |T(f,g)| &\le [T(f,f)]^{\frac{1}{2}}\,[T(g,g)]^{\frac{1}{2}}, && \text{Dragomir [8]} \\ & && \text{for } f,g \in L_2[a,b]; \\ &= \tfrac{\Gamma-\gamma}{2}\,[T(f,f)]^{\frac{1}{2}}, && \text{Matić et al. [11]} \\ & && \text{for } \gamma \le g(t) \le \Gamma,\ t\in[a,b]; \\ &= \tfrac{(\Gamma-\gamma)(\Phi-\phi)}{4} && \text{Grüss (see [12, pp. 295-310])} \\ & && \phi \le f \le \Phi,\ t \in [a,b]. \end{aligned} \tag{2.3}$$

Dragomir [8] obtains numerous results if either f or g or both are known, although the first inequality in (2.3) has a long history (see for example [12], pp. 295-310). The inequalities in (2.3) when proceeding from top to bottom are in the order of increasing coarseness.

Theorem 3. *Let the mapping $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous. Then the following inequality holds. Namely,*

$$
\begin{aligned}
(2.4)\qquad & \left|\tau_n(x) - (-1)^n U_n(x) S_{n-1}(f;a,b)\right| \\
& \le (b-a)\kappa_n(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - S_{n-1}^2(f;a,b)\right]^{\frac{1}{2}}, \quad f^{(n)} \in L_2[a,b], \\
& \le (b-a)\kappa_n(x)\left(\frac{\Gamma_n - \gamma_n}{2}\right), \quad \gamma_n \le f^{(n)}(t) \le \Gamma_n,\ t \in [a,b], \\
& \le (b-a)\frac{(\Phi_n(x) - \phi_n(x))}{4}(\Gamma_n - \gamma_n), \\
& \qquad\qquad \phi_n(x) \le K_n(x,t) \le \Phi_n(x),\ t \in [a,b],
\end{aligned}
$$

where $\tau_n(x)$ is as defined in (1.6),

$$
(2.5)\qquad U_n(x) = \sum_{k=0}^{m}(-1)^{m-k}\left[P_{n,m,k}(x) - P_{n,m,k}(a) + Q_{n,m,k}(b) - Q_{n,m,k}(x)\right],
$$

$$
(2.6)\qquad P_{n,m,k}(t) = r_{n-(m-k)}(t)\, s_k(t), \quad Q_{n,m,k}(t) = u_{n-(m-k)}(t)\, v_k(t)
$$

with $r_.(\cdot)$, $s_.(\cdot)$, $u_.(\cdot)$, $v_.(\cdot)$ satisfying (1.1),

$$
(2.7)\qquad S_n(f;a,b) = \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a},
$$

and

$$
(2.8)\qquad \kappa_n(x) = \left[\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt - \left(\frac{U_n(x)}{b-a}\right)^2\right]^{\frac{1}{2}},
$$

with $K_n(x,t)$ being as defined by (1.2).

Proof. Associating $f(t)$ with $(-1)^n K_n(x,t)$ and $g(t)$ with $f^{(n)}(t)$, then from (1.3), (2.1) and (2.2), we obtain

$$
\begin{aligned}
& T\left((-1)^n K_n(x,\cdot), f^{(n)}(\cdot)\right) \\
= \;& \mathfrak{M}\left((-1)^n K_n(x,\cdot), f^{(n)}(\cdot)\right) - \mathfrak{M}\left((-1)^n K_n(x,\cdot)\right)\mathfrak{M}\left(f^{(n)}(\cdot)\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
(2.9)\qquad & (b-a)\,T\left((-1)^n K_n(x,\cdot), f^{(n)}(\cdot)\right) \\
= \;& \tau_n(x) - (-1)^n U_n(x)\left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right),
\end{aligned}
$$

where $\tau_n(x)$ is the left hand side of (1.6) and

$$
(2.10)\qquad U_n(x) = \int_a^x p_n(t)\,dt + \int_x^b q_n(t)\,dt.
$$

Now, using (1.2)

$$
\int_a^x p_n(t)\,dt = \int_a^x r_{n-m}(t)\, s_m(t)\,dt
$$

from which repeated integration by parts and using the fact that $r.(\cdot)$ and $s.(\cdot)$ satisfy (1.1) gives on using (2.6)

$$\int_a^x p_n(t)\,dt = \sum_{k=0}^{m} (-1)^{m-k} \left[P_{n,m,k}(t)\right]_a^x . \tag{2.11}$$

A similar argument gives on using (2.9)

$$\int_a^b q_n(t)\,dt = \sum_{k=0}^{m} (-1)^{m-k} \left[Q_{n,m,k}(t)\right]_x^b \tag{2.12}$$

and so combining (2.11) and (2.12) in (2.9) gives (2.5).

Now for the bounds in (2.4).

From (2.1) we have, for $K_n(x,t)$ as defined by (1.2),

$$\begin{aligned} &T\left((-1)^n K_n(x,\cdot),(-1)^n K_n(x,\cdot)\right) \\ &= \left[\mathfrak{M}\left(K_n^2(x,\cdot)\right) - \mathfrak{M}^2\left(K_n(x,\cdot)\right)\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\right)^2\right]^{\frac{1}{2}} \\ &= \kappa_n(x), \end{aligned} \tag{2.13}$$

as defined in (2.8) since $U_n(x) = \int_a^b K_n(x,t)\,dt$.

Further, using (2.1), (2.4) and (2.7) gives

$$\begin{aligned} T\left(f^{(n)}(t), f^{(n)}(t)\right) &= \left[\mathfrak{M}\left(\left[f^{(n)}(t)\right]^2\right) - \mathfrak{M}^2\left(f^{(n)}(t)\right)\right]^{\frac{1}{2}} \\ &= \left\{\frac{1}{b-a}\int_a^b \left[f^{(n)}(t)\right]^2 dt - \left[\frac{\int_a^b f^{(n)}(t)\,dt}{b-a}\right]^2\right\}^{\frac{1}{2}} \\ &= \left\{\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - S_{n-1}^2(f;a,b)\right\}^{\frac{1}{2}} \end{aligned}$$

and so combining the above result with (2.13) produces the first inequality.

For $f^{(n)} \in L_\infty[a,b]$ ($\subset L_2[a,b]$ with strict inclusion) then

$$\begin{aligned} 0 &\le T\left(f^{(n)}(t), f^{(n)}(t)\right) \\ &= \frac{1}{b-a}\int_a^b \left|f^{(n)}(t)\right|^2 dt - \left[\frac{\int_a^b f^{(n)}(t)\,dt}{b-a}\right]^2 \\ &\le \left(\frac{\Gamma_n - \gamma_n}{2}\right)^2, \quad \text{where } \gamma_n \le f^{(n)}(t) \le \Gamma_n,\ t \in [a,b], \end{aligned} \tag{2.14}$$

and the third inequality in (2.5), is due to Grüss.

Hence the second inequality in (2.4) is obtained which is coarser than the first.

Further, from (2.13)

$$0 \le \kappa_n^2 \le \left(\frac{\Phi_n(x) - \phi_n(x)}{2}\right)^2, \quad \text{where } \phi_n(x) \le K_n(x,t) \le \Phi_n(x),\ t \in [a,b],$$

which, when combined with (2.14), produces the third bound in (2.4) which is the coarsest bound of all. The proof of the theorem is thus complete. ∎

Remark 1. *The second result in (2.4) is a generalisation of a result by Pearce et al.* [13] *in which each of the branches of the Peano kernel (1.2) satisfy (1.1). That is,*

$$K_n^*(x,t) = \begin{cases} P_n(t), & t \in [a,x]; \\ Q_n(t), & t \in (x,b], \end{cases} \tag{2.15}$$

where $P_n(t)$ and $Q_n(t)$ satisfy (1.2). With the Peano kernel (2.15), they obtained the result

Theorem 4. *Assume that $f : [a,b] \to \mathbb{R}$ is such that $f^{(n)}$ is integrable and $\gamma_n \le f^{(n)} \le \Gamma_n$ for all $t \in [a,b]$. Put*

$$U_n(x) := \frac{1}{b-a}\left[Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)\right]. \tag{2.16}$$

Then for all $x \in [a,b]$, we have the inequality

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right. \\ &\qquad \left. - P_k(a) f^{(k-1)}(a) \Big] - (-1)^n U_n(x) \Big[f^{(n-1)}(b) - f^{(n-1)}(a) \Big] \right| \\ &\qquad\qquad \le \frac{1}{2} K(x) (\Gamma_n - \gamma_n)(b-a), \end{aligned} \tag{2.17}$$

where

$$K(x) := \left\{ \frac{1}{b-a} \int_a^x P_n^2(t)\,dt + \int_x^b Q_n^2(t)\,dt - [U_n(x)]^2 \right\}^{\frac{1}{2}}. \tag{2.18}$$

In the recent paper [8], Dragomir proved the following refinement of (2.17).

Theorem 5. *Assume that the mapping $f : [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_2[a,b]$ $(n \ge 1)$. If we denote*

$$\left[f^{(n-1)}; a, b \right] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \right. \\ &\quad - P_k(a) f^{(k-1)}(a) \Big] - (-1)^n \left[Q_{n+1}(b) - Q_{n+1}(x) \right. \\ &\quad \left. + P_{n+1}(x) - P_{n+1}(a) \right] \left[f^{(n-1)}; a, b \right] \Big| \\ \le\ & K(x)(b-a) \left[\frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left(\left[f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}} \\ & \left(\le \frac{1}{2} K(x)(b-a)(\Gamma_n - \gamma_n) \quad \text{if } \ f^{(n)} \in L_\infty(a,b) \right), \end{aligned} \tag{2.19}$$

for all $x \in [a,b]$ and $K(x)$ as is given in (2.18).

The results of Theorem 3 are generalisations of these which allow each of the branches themselves to be made up of products of functions satisfying (1.1).

Corollary 3. *Let the conditions on f of Theorem 3 hold and let $\alpha,\ x,\ \beta \in [a,b]$ with $a \le \alpha,\ x < \beta \le b$ then,*

$$\left|\binom{n}{m}\tau_n^C(x) - (-1)^n U_n^C(x) S_{n-1}(f;a,b)\right| \tag{2.20}$$
$$\le (b-a)\kappa_n^C(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - S_{n-1}^2(f;a,b)\right]^{\frac{1}{2}},\quad f^{(n)} \in L_2[a,b],$$
$$\le (b-a)\kappa_n^C(x)\left(\frac{\Gamma_n-\gamma_n}{2}\right),\quad \gamma_n \le f^{(n)}(t) \le \Gamma_n,\ t\in[a,b],$$
$$\le (b-a)\frac{\left(\Phi_n^C(x)-\phi_n^C(x)\right)(\Gamma_n-\gamma_n)}{4},$$
$$\phi_n^C(x) \le \kappa_n^C(x,t) \le \Phi_n^C(x),\ t\in[a,b],$$

where $\tau_n^C(x)$ and $S_n(f;a,b)$ are as defined by (1.10) and (2.7) respectively,

$$K_n^C(x,t) = \begin{cases} p_n^C(t) = \dfrac{(t-a)^{n-m}}{(n-m)!}\cdot\dfrac{(t-\alpha)^m}{m!}, & t\in[a,x] \\[2ex] q_n^C(t) = \dfrac{(t-b)^{n-m}}{(n-m)!}\cdot\dfrac{(t-\beta)^m}{m!}, & t\in(x,b], \end{cases} \tag{2.21}$$

$$U_n^C(x) = \frac{1}{(n-m)!m!}\left[\tilde{B}_a(n,m) + \tilde{B}_b(n,m)\right], \tag{2.22}$$

with

$$\tilde{B}_a(n,m) = \begin{cases} \dfrac{(x-a)^{n+1}}{n+1}, & \alpha = a \\[2ex] (-1)^m(\alpha-a)^{n+1}B\left(n-m+1,m+1,\dfrac{x-a}{\alpha-a}\right), & \alpha \ne a \end{cases}$$

$$\tilde{B}_b(n,m) = \begin{cases} \dfrac{(-1)^n(b-x)^{n+1}}{n+1}, & \beta = b \\[2ex] (-1)^{n-m}(b-\beta)^{n+1}B\left(n-m+1,m+1,\frac{b-x}{b-\beta}\right), & \beta \ne b \end{cases}$$

and

$$B(j,k,X) = \int_0^X u^{j-1}(1-u)^{k-1}\,du,\quad \textit{the incomplete beta function,}$$

$$\kappa_n^C(x) = \left\{\frac{1}{(n-m)!m!(b-a)}\left[\tilde{B}_a(2n,2m) + \tilde{B}_b(2n,2m)\right] - \left(\frac{U_n^C(x)}{b-a}\right)^2\right\}^{\frac{1}{2}} \tag{2.23}$$

and

$$\phi_n^C(x) = \min\left\{\phi_n^a(x), \phi_n^b(x)\right\}, \quad \Phi_n^C(x) = \max\left\{\Phi_n^a(x), \Phi_n^b(x)\right\}, \tag{2.24}$$

with

$$\begin{aligned}\phi_n^a(x) &= \inf_{t\in[a,x]} p_n^C(t), \ \phi_n^b(x) = \inf_{t\in(x,b]} q_n^C(t), \\ \Phi_n^a(x) &= \sup_{t\in[a,x]} p_n^C(t), \ \Phi_n^b(x) = \sup_{t\in(x,b]} q_n^C(t).\end{aligned}$$

Proof. Let $r_\cdot^C(\cdot)$, $s_\cdot^C(\cdot)$, $u_\cdot^C(\cdot)$ and $v_\cdot^C(\cdot)$ be as defined in (**??**) giving from (1.16) and (1.2) $K_n^C(x,t)$ as shown in (2.21).

Now,

$$U_n^C(x) = \int_a^b K_n^C(x,t)\,dt = \int_a^x p_n^C(t)\,dt + \int_x^b q_n^C(t)\,dt \tag{2.25}$$

and so using (2.21) consider

$$\begin{aligned}(n-m)!m!\int_a^x p_n^C(t)\,dt &= \int_a^x (t-a)^{n-m}(t-\alpha)^m\,dt \\ &= \frac{(x-a)^{n+1}}{n+1}, \ \alpha = a.\end{aligned} \tag{2.26}$$

If $\alpha \neq a$, let $(\alpha-a)u = t-a$, then

$$\int_a^x (t-a)^{n-m}(t-\alpha)^m\,dt = (-1)^n(\alpha-a)^{n+1}\int_0^{\frac{x-a}{\alpha-a}} u^{n-m}(1-u)^m\,du.$$

Similarly,

$$\begin{aligned}(n-m)!m!\int_x^b q_n^C(t)\,dt &= \int_x^b (t-b)^{n-m}(t-\beta)^m\,dt \\ &= \frac{(-1)^n(b-x)^{n+1}}{n+1}, \ \beta = b.\end{aligned} \tag{2.27}$$

If $\beta \neq b$, let $(b-\beta)v = b-t$ to give

$$\int_x^b (t-b)^{n-m}(t-\beta)^m\,dt = (-1)^{n-m}(b-\beta)^{n+1}\int_0^{\frac{b-x}{b-\beta}} v^{n-m}(1-v)^m\,dv.$$

Substitution of the above results into (2.25) gives (2.22).

Further, from (2.21),

$$\begin{aligned}\int_a^b \left[K_n^C(x,t)\right]^2 dt &= \frac{1}{(n-m)!m!}\left\{\int_a^x (t-a)^{2(n-m)}(t-\alpha)^{2m}\,dt\right. \\ &\quad \left. + \int_x^b (t-b)^{2(n-m)}(t-\beta)^{2m}\,dt\right\},\end{aligned}$$

which, on following a similar procedure to the above gives

$$\int_a^b \left[K_n^C(x,t)\right]^2 dt = \frac{1}{(n-m)!m!}\left[\tilde{B}_a(2n,2m) + \tilde{B}_b(2n,2m)\right]. \tag{2.28}$$

Hence, since

$$\kappa_n^C(x)=\left[\frac{1}{b-a}\int_a^b\left[K_n^C(x,t)\right]^2dt-\left(\frac{U_n^C(x)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{2.29}$$

then from (2.28) and (2.22) we obtain (2.23).

For the first inequality we observe that, from (2.21),

$$\Phi_n^C(x)=\sup_{t\in[a,b]}K_n^C(x,t)=\max\left\{\sup_{t\in[a,x]}p_n^C(t),\sup_{t\in(x,b]}q_n^C(t)\right\}$$

and

$$\phi_n^C(x)=\inf_{t\in[a,b]}K_n^C(x,t)=\min\left\{\inf_{t\in[a,x]}p_n^C(t),\inf_{t\in(x,b]}q_n^C(t)\right\}.$$

The corollary is thus completely proven where the inequalities are in increasing coarseness as discussed in Theorem 3. ∎

Remark 2. *If $\alpha=a$ and $\beta=b$, then a perturbed generalised Ostrowski type result is obtained. If $\alpha=\beta=x$, perturbed trapezoidal type results are obtained for n-time differentiable functions.*

Corollary 4. *Let the conditions of Theorem 3 hold and let $\alpha,\ x,\ \beta\in[a,b]$ with $a\le\alpha,\ x,\ \beta\le b$, then*

$$\begin{aligned}\left|\binom{n}{m}\tau_n^{C^*}(x)-(-1)^nU_n^{C^*}(x)S_{n-1}(f;a,b)\right|&\\ \le(b-a)\kappa_n^{C^*}(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-S_{n-1}^2(f;a,b)\right]^{\frac{1}{2}},&\quad f^{(n)}\in L_2[a,b],\\ \le(b-a)\kappa_n^{C^*}(x)\left(\frac{\Gamma_n-\gamma_n}{2}\right),&\quad\gamma_n\le f^{(n)}(t)\le\Gamma_n,\ t\in[a,b],\\ \le(b-a)\frac{\left(\Phi_n^{C^*}(x)-\phi_n^{C^*}(x)\right)(\Gamma_n-\gamma_n)}{4},&\\ \phi_n^{C^*}(x)\le K_n^{C^*}(x,t)\le\Phi_n^{C^*}(x),&\ t\in[a,b],\end{aligned}\tag{2.30}$$

where $\tau_n^{C^*}(x)$ and $S_n(f;a,b)$ are as defined by (1.17) and (2.7) respectively,

$$K_n^{C^*}(x,t)=\begin{cases}p_n^{C^*}(t)=\dfrac{(t-a)^{n-m}}{(n-m)!}\cdot\dfrac{(t-(x\lambda+(1-\lambda)a))^m}{m!}, & t\in[a,x]\\[2ex] q_n^{C^*}(t)=\dfrac{(t-b)^{n-m}}{(n-m)!}\cdot\dfrac{(t-(x\lambda+(1-\lambda)b))^m}{m!}, & t\in(x,b],\end{cases}\tag{2.31}$$

$$U_n^{C^*}(x)=\frac{\tilde{B}^*(n,m)}{(n-m)!m!}\left[(x-a)^{n+1}+(-1)^n(b-x)^{n+1}\right],\tag{2.32}$$

with

$$\tilde{B}^{*}(n,m)=\begin{cases}\dfrac{1}{n+1}, & \lambda=0,\\ (-1)^{m}\lambda^{n+1}B\left(n-m+1,m+1,\frac{1}{\lambda}\right), & \lambda\neq 0,\end{cases}$$

and

$$B(j,k,X)=\int_0^X u^{j-1}(1-u)^{k-1}\,du,\quad \textit{the incomplete beta function,}$$

$$\text{(2.33)}\qquad \kappa_n^{C^*}(x) = \left\{\frac{\lambda^{2n+1}\tilde{B}^{*}(n,m)}{(n-m)!m!(b-a)}\left[(x-a)^{2n+1} + (b-x)^{2n+1}\right]-\left(\frac{U_n^{C^*}(x)}{b-a}\right)^2\right\}^{\frac{1}{2}}$$

and

$$\phi_n^{C^*}(x) = \min\left\{\inf_{t\in[a,x]}p_n^{C^*}(t),\inf_{t\in(x,b]}q_n^{C^*}(t)\right\},$$

$$\Phi_n^{C^*}(x) = \max\left\{\sup_{t\in[a,x]}p_n^{C^*}(t),\sup_{t\in(x,b]}q_n^{C^*}(t)\right\}.$$

Proof. Taking $\alpha=\lambda x+(1-\lambda)a$ and $\beta=\lambda x+(1-\lambda)b$ in Corollary 3 produces the above results after some simplification. ∎

Remark 3. *Taking $\lambda=0$ in Corollary 4 implying that $\alpha=a$ and $\beta=b$ produces a perturbed Ostrowski type results. If $\lambda=1$, then perturbed trapezoidal type results are obtained for $n-$time differentiable functions.*

Corollary 5. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then, for $\alpha,\ x,\ \beta\in[a,b]$ with $a\leq\alpha,\ x,\ \beta\leq b$, the perturbed Simpson rule*

$$\text{(2.34)}\qquad \left|\tau_n^S(x)-\frac{(-1)^n}{n}U_n^S(x)S_{n-1}(f;a,b)\right|$$
$$\leq\frac{(b-a)}{n}\kappa_n^S(x)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-S_{n-1}^2(f;a,b)\right]^{\frac{1}{2}},\quad f^{(n)}\in L_2[a,b],$$
$$\leq\frac{(b-a)}{n}\kappa_n^S(x)\left(\frac{\Gamma_n-\gamma_n}{2}\right),\quad \gamma_n\leq f^{(n)}(t)\leq\Gamma_n,\ t\in[a,b],$$
$$\leq\frac{(b-a)}{n}\cdot\frac{\left(\Phi_n^S(x)-\phi_n^S(x)\right)(\Gamma_n-\gamma_n)}{4},$$
$$\phi_n^S(x)\leq K_n^S(x,t)\leq\Phi_n^S(x),\ t\in[a,b],$$

where $\tau_n^S(x)$ *and* $S_n(f;a,b)$ *are as defined by (1.24) and (2.7) respectively,*

$$K_n^S(x,t)=\begin{cases} p_n^S(t)=\dfrac{(t-a)^{n-1}}{(n-1)!}(t-\alpha_n(x)), \\ \\ q_n^S(t)=\dfrac{(t-b)^{n-1}}{(n-1)!}(t-\beta_n(x)), \end{cases} \tag{2.35}$$

with

$$\alpha_n(x)=\lambda_n x+(1-\lambda_n)a, \quad \beta_n(x)=\lambda_n x+(1-\lambda_n)b, \quad \lambda_n=\frac{n}{3}, \tag{2.36}$$

$$U_n^S(x)=\frac{n(1-\lambda_n)-\lambda_n}{(n+1)!}\left[(x-a)^{n+1}+(-1)^n(b-x)^{n+1}\right], \tag{2.37}$$

(2.38)

$$\kappa_n^S(x)=\left\{\frac{\lambda^{2n+1}B\left(2n-1,3,\frac{1}{\lambda}\right)}{(n-1)!(b-a)}\left[(x-a)^{2n+1}+(b-x)^{2n+1}\right]-\left(\frac{U_n^S(x)}{b-a}\right)^2\right\}^{\frac{1}{2}}$$

and

$$\begin{aligned} \phi_n^S(x) &= \frac{1}{(n-1)!}\min\left\{\inf_{U\in[0,x-a]}X(U), \inf_{V\in(0,b-x]}Y(V)\right\}, \\ \Phi_n^S(x) &= \frac{1}{(n-1)!}\min\left\{\sup_{U\in[0,x-a]}X(U), \sup_{V\in(0,b-x]}Y(V)\right\}, \end{aligned} \tag{2.39}$$

with

$$X(U)=U^{n-1}(U-\lambda_n(x-a)),\ Y(V)=(-1)^n V^{n-1}(V-\lambda_n(b-x)).$$

Proof. Taking $m=1$ in Corollary 4 and (2.22) and (2.36) in place of (2.31) gives

$$U_n^S(x)=\int_a^x p_n^S(t)\,dt+\int_x^b q_n^S(t)\,dt. \tag{2.40}$$

Taking $m=1$ and $\lambda\equiv\lambda_n^S$ in (2.32) gives $U_n^S(x)$. Alternatively, direct calculation gives,

$$\begin{aligned} \int_a^x p_n^S(t)\,dt &= \int_a^x \frac{(t-a)^{n-1}}{(n-1)!}(t-\alpha_n(x))\,dt \\ &= \frac{(t-a)^n}{(n+1)!}\left[(n+1)(t-\alpha_n(x))-(t-a)\right]\Big]_{t=a}^{x} \\ &= \frac{(x-a)^n}{(n+1)!}\left[(n+1)(x-\alpha_n(x))-(x-a)\right] \\ &= \frac{(x-a)^n}{(n+1)!}\left[(n+1)(1-\lambda_n)-1\right] \\ &= \frac{(x-a)^n}{(n+1)!}\left[n(1-\lambda_n)-\lambda_n\right]. \end{aligned}$$

Similarly,

$$\begin{aligned}\int_x^b q_n^S(t)\,dt &= \int_x^b \frac{(t-b)^{n-1}}{(n-1)!}\left(t-\beta_n(x)\right)dt \\ &= (-1)^n \frac{(b-x)^n}{(n+1)!}\left[(n)(1-\lambda_n)-\lambda_n\right].\end{aligned}$$

Combining the above results into (1.17) produces (2.37).

Now, to determine $\kappa_n^S(x)$ from (2.29) we have, using (2.35)

$$\frac{1}{b-a}\int_a^b \left[K_n^S(x,t)\right]^2 dt = \frac{1}{b-a}\left[\int_a^x \left(p_n^S(t)\right)^2 dt + \int_x^b \left(q_n^S(t)\right)^2 dt\right].$$

For direct calculation we require integration by parts twice. Alternatively, utilising (2.23) with $m=1$ and $\alpha_n(x)$, $\beta_n(x)$ being as given by (2.36) produces the stated result (2.38).

Now,

$$\Phi_n^S(x) = \max\left\{\sup_{t\in[a,x]} p_n^S(t), \sup_{t\in(x,b]} q_n^S(t)\right\},$$

and on using (2.36),

$$\begin{aligned}\sup_{t\in[a,x]} p_n^S(t) &= \frac{1}{(n-1)!}\sup_{U\in[0,x-a]} U^{n-1}\left(U-\lambda_n(x-a)\right), \\ \sup_{t\in(x,b]} q_n^S(t) &= \frac{1}{(n-1)!}\sup_{V\in[0,b-x)} (-1)^n V^{n-1}\left(V-\lambda_n(b-x)\right).\end{aligned}$$

In a similar fashion

$$\phi_n^S(x) = \min\left\{\inf_{t\in[a,x]} p_n^S(t), \inf_{t\in(x,b]} q_n^S(t)\right\},$$

and using the above expressions with sup replaced by inf gives the results as stated in (2.39).

The proof is now complete. ∎

Remark 4. *The perturbed results obtained above through the use of the Chebychev functional (2.1) and the resulting bounds given by (2.3) may be advantageous when compared to the first bounds in (1.6), (1.10), (1.17) and (1.23). For functions $g,h:[a,b]\to\mathbb{R}$ and $\gamma\le g(t)\le\Gamma$, then $\frac{\Gamma-\gamma}{2}\le\|g\|_\infty$. It is however, difficult to compare $\|h\|_1\|g\|_\infty$ obtained for the unperturbed results of previous sections, with the perturbed bounds of the form*

$$(b-a)\|h\|_\infty\sigma^{\frac{1}{2}}(g) < (b-a)\|h\|_\infty\frac{\Gamma-\gamma}{2} < (b-a)\frac{(\Phi-\phi)(\Gamma-\gamma)}{2},$$

where

$$\begin{aligned}\sigma(g) &= \frac{1}{b-a}\|g\|_2^2 - S^2(g;a,b),\ S(g;a,b)=\frac{g(a)-g(b)}{b-a} \\ \text{and } \phi &\le h(t)\le\Phi.\end{aligned}$$

This is so since a comparison between $\|h\|_1$ and $\|h\|_\infty$ cannot readily be made.

Remark 5. *In a recent article Cerone and Dragomir* [3] *obtained the following results of Grüss type for the Chebychev functional* $T(f,g)$*. They utilised the notion of a* $\Delta-$*seminorm introduced by Cerone and Dragomir* [2] *where*

$$\text{(2.41)}\quad \begin{cases} \|f\|_p^{\Delta} := \left(\int_a^b \int_a^b |f(s)-f(t)|^p \, dsdt\right)^{\frac{1}{p}}, & \text{for } f \in L_p[a,b],\ p \in [1,\infty), \\ \text{and} & \\ \|f\|_\infty^{\Delta} := ess \sup\limits_{(s,t)\in[a,b]^2} |f(s)-f(t)|, & \text{for } f \in L_\infty[a,b]. \end{cases}$$

If we consider $f_\Delta : [a,b]^2 \to \mathbb{R}$*, where* $f_\Delta(s,t) = f(s) - f(t)$*, then*

$$\text{(2.42)}\qquad \|f\|_p^{\Delta} \equiv \|f_\Delta\|_p, \quad p \in [1,\infty]$$

with $\|\cdot\|_p$ *being the usual Lebesgue* $p-$*norms on* $[a,b]^2$*.*

The following theorem giving bounds for the Chebychev functional in terms of $\Delta-$seminorms (2.41) holds (see also Cerone and Dragomir [3]):

Let $f,g : [a,b] \to \mathbb{R}$ *be measurable on* $[a,b]$*. Then the inequality*

$$\text{(2.43)}\qquad |T(f,g)| \le \frac{1}{2(b-a)^2} \|f\|_p^{\Delta} \, \|g\|_q^{\Delta}$$

holds provided the integrals exist, where $T(f,g)$ *is the Chebychev functional given by (2.1) – (2.2),* $p=1$*,* $q=\infty$ *or* $q=1$*,* $p=\infty$ *or* $p>1$*,* $\frac{1}{p}+\frac{1}{q}=1$*, and* $\|f\|_{\cdot}^{\Delta}$ *is defined by (2.41).*

Using the fact that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous then

$$f(s) - f(t) = \int_t^s f'(u)\, du,$$

the following theorem was obtained by Cerone and Dragomir [2]:

For $f : [a,b] \to \mathbb{R}$ *absolutely continuous on* $[a,b]$ *the following inequalities hold.*

(i) *If* $p \in [1,\infty)$*, then*

$$\text{(2.44)}\qquad \|f\|_p^{\Delta} \le \begin{cases} \dfrac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty, & f' \in L_\infty[a,b], \\[2ex] \dfrac{(2\delta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\delta}+\frac{2}{p}}}{[(p+\delta)(p+2\delta)]^{\frac{1}{p}}} \|f'\|_\gamma, & f' \in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ (b-a)^{\frac{2}{p}} \|f'\|_1, & \end{cases}$$

(ii)

$$\text{(2.45)}\qquad \|f\|_\infty^{\Delta} \le \begin{cases} (b-a)\|f'\|_\infty, & f' \in L_\infty[a,b]; \\[1ex] (b-a)^{\frac{1}{\delta}} \|f'\|_\gamma, & f' \in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|f'\|_1. & \end{cases}$$

Note that, if $p = q = 2$ is taken in Theorem 2 then the first result in (2.3) is obtained. Both of the above two theorems may be used to obtain bounds on the perturbed rules that would result if we associated the Peano kernel $K_n(x,\cdot)$ with $f(\cdot)$ and $f^{(n)}(\cdot)$ with $g(\cdot)$. This however will not be pursued further here.

3. Applications in Numerical Integration

Any of the inequalities in the earlier sections may utilised for numerical implementation. Here we illustrate the procedure.

Consider the partition $I_r : a = x_0 < x_1 < \cdots < x_{r-1} < x_r = b$ of the interval $[a,b]$ and let the intermediate points $\boldsymbol{\xi} = \left(\xi_0, \ldots, \xi_{r-1}\right)$ where $\xi_j \in [x_j, x_{j+1}]$ for $j = 0, 1, \ldots, r-1$. Define the formula

$$\begin{aligned} &\mathcal{A}_{r,n}(f, I_r, \boldsymbol{\xi}) \\ = \quad &\frac{1}{\binom{n}{m}} \sum_{j=0}^{r-1} \sum_{k=0}^{n-1} (-1)^{k+1} \left\{ \left[p_n^{(k)}(\xi_j) - q_n^{(k)}(\xi_j) \right] f^{(n-1-k)}(\xi_j) \right. \\ &\left. + q_n^{(k)}(x_{j+1}) f^{(n-1-k)}(x_{j+1}) - p_n^{(k)}(x_j) f^{(n-1-k)}(x_j) \right\}, \end{aligned} \tag{3.1}$$

which may be rewritten in a form that reduces the number of function evaluations, and thus a more efficient form, as

$$\begin{aligned} &\mathcal{A}_{r,n}(f, I_r, \boldsymbol{\xi}) \\ = \quad &\frac{1}{\binom{n}{m}} \sum_{k=0}^{n-1} (-1)^{k+1} \left\{ q_n^{(k)}(x_r) f^{(n-1-k)}(x_r) \right. \\ &- p_n^{(k)}(x_0) f^{(n-1-k)}(x_0) + \left[p_n^{(k)}(\xi_0) - q_n^{(k)}(\xi_0) \right] f^{(n-1-k)}(\xi_0) \\ &+ \sum_{j=0}^{r-1} \left\{ \left[p_n^{(k)}(\xi_j) - q_n^{(k)}(\xi_j) \right] f^{(n-1-k)}(\xi_j) \right. \\ &\left.\left. + \left[q_n^{(k)}(x_j) - p_n^{(k)}(x_j) \right] f^{(n-1-k)}(x_j) \right\} \right\}, \end{aligned} \tag{3.2}$$

where $p_n^{(k)}(\cdot)$ and $q_n^{(k)}(\cdot)$ are given by (1.4).

The following theorem holds involving $\mathcal{A}_{r,n}(f, I_r, \boldsymbol{\xi})$ as given by either (3.1) or (3.2).

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and I_r is a partition of $[a,b]$ as described above. The following quadrature rule then holds*

$$\int_a^b f(x)\,dx = \mathcal{A}_{r,n}(f, I_r, \boldsymbol{\xi}) + \mathcal{R}_{r,n}(f, I_r, \boldsymbol{\xi}), \tag{3.3}$$

where $\mathcal{A}_{r,n}$ is as defined by (3.1) (or (3.2)) and the remainder $\mathcal{R}_{r,n}(f, I_r, \boldsymbol{\xi})$ satisfies the estimation

$$\binom{n}{m}\left|\mathcal{R}_{r,n}(f, I_r, \boldsymbol{\xi})\right| \leq \begin{cases} \left\|f^{(n)}\right\|_{\infty} \sum_{j=0}^{r-1} B_n(1, \xi_j), & \text{for } f^{(n)} \in L_{\infty}[a,b]; \\ \left\|f^{(n)}\right\|_{p} \sum_{j=0}^{r-1} \left[B_n(q, \xi_j)\right]^{\frac{1}{q}}, & \text{for } f^{(n)} \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left\|f^{(n)}\right\|_{1} \max_{j=0,\ldots,r-1} \theta_n(\xi_j), & \text{for } f^{(n)} \in L_1[a,b], \end{cases} \tag{3.4}$$

where

$$B_n(q, \xi_j) = \int_{x_j}^{\xi_i} |p_n(x)|^q\, dx + \int_{\xi_j}^{x_{j+1}} |q_n(x)|^q\, dx,$$

$$\theta_n(\xi_j) = \frac{M_n(x_j, \xi_j) + M_n(\xi_j, x_{j+1})}{2} + \left|\frac{M_n(\xi_j, x_{j+1}) - M_n(x_j, \xi_j)}{2}\right|, \tag{3.5}$$

$$M_n(x_j, \xi_j) = \sup_{x \in [x_j, \xi_j]} |p_n(x)|, \quad M_n(\xi_j, x_{j+1}) = \sup_{x \in [\xi_j, x_{j+1}]} |q_n(x)|, \tag{3.6}$$

and $p_n(\cdot)$, $q_n(\cdot)$ are as defined in (1.2), $p_n^{(k)}(\cdot)$, $q_n^{(k)}(\cdot)$ are given by (1.4).

Proof. Applying the results of Theorem 1 on the interval $[x_j, x_{j+1}]$ gives from (1.6)

$$\begin{aligned} \left|\tau_n(\xi_j)\right| \; : \; &= \Bigg|\binom{n}{m}\int_{x_j}^{x_{j+1}} f(x)\,dx - \sum_{k=0}^{n-1}(-1)^{k+1} \\ &\quad \times \left\{\left[p_n^{(k)}(\xi_j) - q_n^{(k)}(\xi_j)\right] f^{(n-1-k)}(\xi_j)\right. \\ &\quad + \left. q_n^{(k)}(x_{j+1}) f^{(n-1-k)}(x_{j+1}) - p_n^{(k)}(x_j) f^{(n-1-k)}(x_j)\right\}\Bigg| \\ &\leq \begin{cases} B_n(1, \xi_j) \sup_{x \in [x_j, x_{j+1}]} \left|f^{(n)}(x)\right|, \\ \left[B_n(q, \xi_j)\right]^{\frac{1}{q}} \left[\int_{x_j}^{x_{j+1}} \left|f^{(n)}(x)\right|^p dx\right]^{\frac{1}{p}}, \\ \theta_n(\xi_j) \int_{x_j}^{x_{j+1}} \left|f^{(n)}(x)\right| dx. \end{cases} \end{aligned} \tag{3.7}$$

Summing over j from 0 to $r-1$ and using the generalised triangle inequality gives

$$\binom{n}{m}\left|\mathcal{R}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)\right| \leq \begin{cases} \sum_{j=0}^{r-1} B_n\left(1,\xi_j\right) \sup_{x\in[x_j,x_{j+1}]}\left|f^{(n)}(x)\right|, \\ \sum_{j=0}^{r-1}\left[B_n\left(q,\xi_j\right)\right]^{\frac{1}{q}}\left[\int_{x_j}^{x_{j+1}}\left|f^{(n)}(x)\right|^p dx\right]^{\frac{1}{p}}, \\ \sum_{j=0}^{r-1}\theta_n\left(\xi_j\right)\int_{x_j}^{x_{j+1}}\left|f^{(n)}(x)\right|dx. \end{cases} \tag{3.8}$$

Now, since $\sup_{x\in[x_j,x_{j+1}]}\left|f^{(n)}(x)\right| \leq \left\|f^{(n)}\right\|_\infty$ on the whole interval under consideration, the first inequality in (3.4) readily follows.

Further, using the discrete Hölder inequality, we have

$$\begin{aligned} &\sum_{j=0}^{r-1}\left[B_n\left(q,\xi_j\right)\right]^{\frac{1}{q}}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(x)\right|^p dx\right)^{\frac{1}{p}} \\ \leq\ &\left[\sum_{j=0}^{r-1}B_n\left(q,\xi_j\right)\right]^{\frac{1}{q}}\times\left[\sum_{j=0}^{r-1}\int_{x_j}^{x_{j+1}}\left|f^{(n)}(x)\right|^p dx\right]^{\frac{1}{p}} = \left\|f^{(n)}\right\|_p\sum_{j=0}^{r-1}\left[B_n\left(q,\xi_j\right)\right]^{\frac{1}{q}} \end{aligned}$$

and thus the second inequality in (3.4) is proven on noting the definition of Lebesgue norms (1.9).

Finally, let us observe from (3.8) that

$$\sum_{j=0}^{r-1}\theta_n\left(\xi_j\right)\int_{x_j}^{x_{j+1}}\left|f^{(n)}(x)\right|dx \leq \max_{j=0,\ldots,r-1}\theta_n\left(\xi_j\right)\sum_{j=0}^{r-1}\int_{x_j}^{x_{j+1}}\left|f^{(n)}(x)\right|dx.$$

Hence the theorem is completely proved on using (1.8). ∎

The following theorem demonstrates the implementation of perturbed rules and is based on Theorem 3.

Theorem 7. *Let the conditions of Theorem 6 hold. Then*

$$\int_a^b f(x)\,dx = \mathcal{A}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)+\mathcal{P}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)+\tilde{\mathcal{R}}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right), \tag{3.9}$$

where $\mathcal{A}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)$ *is as defined by (3.1) or (3.2),*

$$\binom{n}{m}\mathcal{P}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right) = (-1)^n\sum_{j=0}^{r-1}U_n\left(\xi_j\right)\cdot S_{n-1}\left(f;x_j;x_{j+1}\right) \tag{3.10}$$

and the remainder $\tilde{\mathcal{R}}_{r,n}(f, I_r, \boldsymbol{\xi})$ *satisfies*

$$
\begin{aligned}
(3.11)\quad & \binom{n}{m}\tilde{\mathcal{R}}_{r,n}(f, I_r, \boldsymbol{\xi}) \\
\leq\ & \left[\sum_{j=0}^{r-1} h_j \kappa_n^2(\xi_j)\right]^{\frac{1}{2}} \left[\left\|f^{(n)}\right\|_2^2 - \sum_{j=0}^{r-1} h_j S_{n-1}^2(f, x_j, x_{j+1})\right]^{\frac{1}{2}}, \quad f^{(n)} \in L_2[a,b] \\
\leq\ & \left[\sum_{j=0}^{r-1} h_j \kappa_n^2(\xi_j)\right]^{\frac{1}{2}} \left[\sum_{j=0}^{r-1}\left(\frac{\Gamma_{n,j} - \gamma_{n,j}}{2}\right)\right]^{\frac{1}{2}}, \\
& \qquad \gamma_{n,j} \leq f^{(n)}(x) \leq \Gamma_{n,j}, \quad x \in [x_j, x_{j+1}]; \\
\leq\ & \left[\sum_{j=0}^{r-1} h_j^2 \left(\frac{\Phi_{n,j}(\xi_j) - \phi_{n,j}(\xi_j)}{2}\right)\right]^{\frac{1}{2}} \left[\sum_{j=0}^{r-1}\left(\frac{\Gamma_{n,j} - \gamma_{n,j}}{2}\right)\right]^{\frac{1}{2}}, \\
& \qquad \phi_{n,j}(\xi_j) \leq K_n(\boldsymbol{\xi}, x) \leq \Phi_{n,j}(\xi_j), \quad x \in [x_j, x_{j+1}],
\end{aligned}
$$

where

$$
\begin{aligned}
U_n(\xi_j) &= \sum_{k=0}^{m} (-1)^{m-k}\left[P_{n,m,k}(\xi_j) - P_{n,m,k}(x_j)\right. \\
&\qquad \left. + Q_{n,m,k}(x_{j+1}) - Q_{n,m,k}(\xi_j)\right], \\
P_{n,m,k}(\cdot) &= r_{n-(m-k)}(\cdot)\, s_k(\cdot), \ Q_{n,m,k}(\cdot) = u_{n-(m-k)} v_k(\cdot)
\end{aligned}
$$

with $r_{\cdot}(\cdot)$, $s_{\cdot}(\cdot)$, $u_{\cdot}(\cdot)$, $v_{\cdot}(\cdot)$ *satisfying (1.1),*

$$S_n(f; x_j, x_{j+1}) = \frac{f^{(n)}(x_{j+1}) - f^{(n)}(x_j)}{h_j}, \quad h_j = x_{j+1} - x_j,$$

and

$$\kappa_n(\xi_j) = \left[h_j^{-1}\int_{x_j}^{x_{j+1}} K_n^2(\xi_j, x)\,dx - \left(\frac{U_n(\xi_j)}{h_j}\right)^2\right]^{\frac{1}{2}},$$

$K_n(\cdot,\cdot)$ *as given by (1.2).*

Proof. Applying the results for Theorem 3 on the interval $[x_j, x_{j+1}]$ produces from (3.7) and the first inequality in (2.4) with $\xi_j \in [x_j, x_{j+1}]$

$$
\begin{aligned}
& \left|\tau_n(\xi_j) - (-1)^n U_n(\xi_j) S_n(f; x_j, x_{j+1})\right| \\
\leq\ & h_j \kappa_n(\xi_j)\left[h_j^{-1}\int_{x_j}^{x_{j+1}}\left[f^{(n)}(x)\right]^2 dx - S_{n-1}^2(f; x_j, x_{j+1})\right]^{\frac{1}{2}}.
\end{aligned}
$$

Summing over j from 0 to $r-1$ produces

$$\begin{aligned}
&\binom{n}{m}\left|\mathcal{R}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)\right| \\
&\leq \sum_{j=0}^{r-1} h_j \kappa_n\left(\xi_j\right)\left[h_j^{-1}\int_{x_j}^{x_{j+1}}\left[f^{(n)}(x)\right]^2 dx - S_{n-1}^2\left(f;x_j,x_{j+1}\right)\right]^{\frac{1}{2}} \\
&= \sum_{j=0}^{r-1} h_j^{\frac{1}{2}} \kappa_n\left(\xi_j\right)\left[\int_{x_j}^{x_{j+1}}\left[f^{(n)}(x)\right]^2 dx - h_j S_{n-1}^2\left(f;x_j,x_{j+1}\right)\right]^{\frac{1}{2}}
\end{aligned}$$

which upon using Hölder's inequality gives

$$\begin{aligned}
&\binom{n}{m}\left|\mathcal{R}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)\right| \\
&\leq \left[\sum_{j=0}^{r-1} h_j \kappa_n^2\left(\xi_j\right)\right]^{\frac{1}{2}} \times \left[\sum_{j=0}^{r-1}\int_{x_j}^{x_{j+1}}\left(f^{(n)}(x)\right)^2 dx - h_j S_{n-1}^2\left(f;x_j,x_{j+1}\right)\right]^{\frac{1}{2}}
\end{aligned} \tag{3.12}$$

producing the first inequality in (3.11) on noting that

$$\sum_{j=0}^{r-1}\int_{x_j}^{x_{j+1}}\left(f^{(n)}(x)\right)^2 dx = \left\|f^{(n)}\right\|_2^2.$$

Now for the second inequality. Using (2.14) from (3.12) gives the coarser inequality

$$\begin{aligned}
&\binom{n}{m}\left|\mathcal{R}_{r,n}\left(f,I_r,\boldsymbol{\xi}\right)\right| \\
&\leq \left[\sum_{j=0}^{r-1} h_j \kappa_n^2\left(\xi_j\right)\right]^{\frac{1}{2}}\left[\sum_{j=0}^{r-1}\left(\frac{\Gamma_{n,j}-\gamma_{n,j}}{2}\right)\right]^{\frac{1}{2}}, \quad \gamma_{n,j} \leq f^{(n)}(x) \leq \Gamma_{n,j}.
\end{aligned}$$

Using (2.14) again produces the third inequality as stated in (3.11). ∎

Remark 6. *We note that the second inequality may be constrained by a coarser bound since*

$$\sum_{j=0}^{r-1}\left(\frac{\Gamma_{n,j}-\gamma_{n,j}}{2}\right) \leq \frac{r}{2}D_n \leq \frac{r}{2}D_n^*,$$

where

$$D_n = \max_{j=0,\ldots,r-1}\left(\Gamma_{n,j}-\gamma_{n,j}\right), \quad \gamma_{n,j} \leq f^{(n)}(x) \leq \Gamma_{n,j},\ x \in \left[x_j,x_{j+1}\right]$$

and

$$D_n^* = \Gamma_n - \gamma_n,\ \gamma_n \leq f^{(n)}(x) \leq \Gamma_n,\ \ x \in [a,b]\,.$$

A similar exercise for the third inequality in (3.11) is somewhat more difficult since there is a dependent on the location of the ξ_j within each interval $\left[x_j,x_{j+1}\right]$.

For a particular partition I_r and interior points $\boldsymbol{\xi}$

$$\sum_{j=0}^{r-1} h_j^2 \left(\frac{\Phi_{n,j}(\xi_j) - \phi_{n,j}(\xi_j)}{2} \right) \leq \frac{\delta_n}{2} \sum_{j=0}^{r-1} h_j^2 \leq \frac{\delta_n^*}{2} \sum_{j=0}^{r-1} h_j^2,$$

where $\delta_n = \max_{j=0,\ldots,r-1} \left(\Phi_{n,j}(\xi_j) - \phi_{n,j}(\xi_j) \right)$, $\phi_{n,j}(\xi_j) \leq K_n(\xi_j, x) \leq \Phi_{n,j}(\xi_j)$, $x \in [x_j, x_{j+1}]$ *and* $\delta_n^* = \sup_{j=0,\ldots,r-1} \Phi_{n,j}(\xi_j) - \inf_{j=0,\ldots,r-1} \phi_{n,j}(\xi_j)$.

4. Concluding Remarks

In the application of the current work to quadrature, if we wished to approximate $\int_a^b f(x)\,dx$ using a rule $Q(f, I_r)$ with bound $E(r)$, where I_r is a uniform partition for example, with an accuracy of $\varepsilon > 0$, then we require $r_\varepsilon \in \mathbb{N}$ where

$$r_\varepsilon \geq \left[E^{-1}(\varepsilon)\right] + 1,$$

with $[w]$ denoting the integer part of $w \in \mathbb{R}$.

The approach thus described enables the user the predetermine a partition required to *assure* the result to be within a certain error tolerance. This approach is somewhat different from that commonly used of systematic mesh refinement followed by comparison of successive approximations forming the basis of a stopping rule.

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