# NEW ESTIMATES OF THE KULLBACK-LEIBLER DISTANCE AND APPLICATIONS

S. S. DRAGOMIR AND V. GLUŠČEVIĆ

Abstract. New estimates of the Kullback-Leibler distance and applications for Shannon's entropy and mutual information are given.

## 1. Introduction

The *relative entropy* is a measure of the distance between two distributions. In statistics, it arises as an expected logarithm of the likelihood ratio. The relative entropy $D(p\|q)$ is a measure of the inefficiency of assuming that the distribution is $q$ when the true distribution is $p$. For example, if we knew the true distribution of the random variable, then we could construct a code with average description length $H(p)$. If, instead, we used the code for a distribution $q$, we would need $H(p)+D(p\|q)$ bits on the average to describe the random variable [1, p. 18].

With ln we will denote the natural logarithm throughout the paper.

**Definition 1.** *The relative entropy or Kullback-Leibler distance between two probability mass functions $p(x)$ and $q(x)$ is defined by*

$$D(p\|q) := \sum_{x\in\mathcal{X}} p(x)\ln\left(\frac{p(x)}{q(x)}\right) = E_p \ln\left(\frac{p(X)}{q(X)}\right).$$

In the above definition, we use the convention (based on continuity arguments) that $0\ln\left(\frac{0}{q}\right)=0$ and $p\ln\left(\frac{p}{0}\right)=\infty$.

It is well-known that relative entropy is always non-negative and equal to zero if and only if $p=q$. However, it is not a true distance between distributions since it is not symmetric and does not satisfy the triangle inequality.

The following theorem is of fundamental importance [1, p. 26].

**Theorem 1.** *(Information Inequality) Let $p(x)$, $q(x)\in\mathcal{X}$, be two probability mass functions. Then*

$$D(p\|q)\geq 0 \tag{1.1}$$

*with equality if and only if*

$$p(x)=q(x) \quad \textit{for all } x\in\mathcal{X}.$$

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Actually, the inequality (1.1) can be improved as follows (see , [1, p. 300]):

**Theorem 2.** *Let $p, q$ be as above. Then*

$$D(p\|q) \geq \frac{1}{2}\|p-q\|_1^2 \tag{1.2}$$

*where* $\|p-q\|_1 = \sum_{x\in\mathcal{X}} |p(x)-q(x)|$ *is the usual* 1*-norm of* $p-q$*. The equality holds if and only if* $p = q$*.*

We remark that the argument of (1.2) is not based on the convexity of the map $-\ln(.)$.

We introduce *mutual information*, which is a measure of the amount of information that one random variable contains about another random variable. It is the reduction in the uncertainty of one random variable due to the knowledge of the other [1, p. 18].

**Definition 2.** *Consider two random variables* $X$ *and* $Y$ *with a joint probability mass function* $p(x,y)$ *and marginal probability mass function* $p(x)$ *and* $q(y)$*. The mutual information is the relative entropy between the joint distribution and the product distribution, i.e.,*

$$I(X;Y) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\ln\left(\frac{p(x,y)}{p(x)q(y)}\right) = D\left(p(x,y) \parallel p(x)q(y)\right)$$
$$= E_{p(x,y)}\ln\left(\frac{p(X,Y)}{p(X)q(Y)}\right).$$

The following corollary of Theorem 1 holds [1, p. 27].

**Corollary 1.** *(Non-negativity of mutual information): For any two random variables,* $X$*,* $Y$ *we have*

$$I(X;Y) \geq 0$$

*with equality if and only if* $X$ *and* $Y$ *are independent.*

We follow with an improvement of this result via Theorem 2.

**Corollary 2.** *For any two random variables,* $X$*,* $Y$ *we have*

$$I(X;Y) \geq \frac{1}{2}\left[\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} |p(x,y)-p(x)q(y)|\right]^2 \geq 0$$

*with equality if and only if* $X$ *and* $Y$ *are independent.*

Now, let $u(x) = \frac{1}{|\mathcal{X}|}$ be the uniform probability mass function on $\mathcal{X}$ and let $p(x)$ be the probability mass function for $X$.

It is well-known that [1, p. 27]

$$D(p\|u) = \sum_{x\in\mathcal{X}} p(x)\ln\frac{p(x)}{u(x)} = \ln|\mathcal{X}| - H(X).$$

The following corollary of Theorem 1 is important [1, p. 27].

**Corollary 3.** *Let $X$ be a random variable and $|\mathcal{X}|$ denotes the number of elements in the range of $X$. Then*

$$H(X) \leq \ln |\mathcal{X}|$$

*with equality if and only if $X$ has a uniform distribution over $\mathcal{X}$.*

Using Theorem 2 we can also state the following.

**Corollary 4.** *Let $X$ be as above. Then*

$$\ln |\mathcal{X}| - H(X) \geq \frac{1}{2} \left[ \sum_{x \in \mathcal{X}} \left| p(x) - \frac{1}{|\mathcal{X}|} \right| \right]^2 \geq 0.$$

*The equality holds if and only if $p$ is uniformly distributed on $\mathcal{X}$ .*

In the recent paper [2], the authors proved between other the following upper bound for the relative entropy and employed it in Coding Theory in connection to Noiseless Coding Theorem:

**Theorem 3.** *Under the above assumptions for $p(x)$ and $q(x)$ we have the inequality*

$$\sum_{x \in \mathcal{X}} \frac{p^2(x)}{q(x)} - 1 \geq D(p\|q)$$

*with equality if and only if $p(x) = q(x)$ for all $x \in \mathcal{X}$ .*

The following upper bound for the mutual information holds.

**Corollary 5.** *For any two random variables, $X$, $Y$ we have*

$$\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \frac{p^2(x,y)}{p(x)q(y)} - 1 \geq I(X;Y)$$

*with equality if and only if $X$ and $Y$ are independent.*

Finally, we note that the following upper bound for the difference $\log |\mathcal{X}| - H(X)$ is valid.

**Corollary 6.** *We have*

$$|\mathcal{X}| \sum_{x \in \mathcal{X}} p^2(x) - 1 \geq \ln |\mathcal{X}| - H(X)$$

*with equality if and only if $p$ is uniformly distributed on $\mathcal{X}$.*

Our aim is to point out some bounds for the relative entropy and to apply them for Shannon's entropy and mutual information.

## 2. **Some Inequalities for the Logarithmic Mapping**

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

**Theorem 4.** *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let $n$ be a positive integer. If $f : I \longrightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on $I$, then for each $x \in I$*

$$f(x) = T_n(f; a, x) + R_n(f; a, x) \tag{2.1}$$

*where $T_n(f; a, x)$ is Taylor's polynomial, i.e.,*

$$T_n(f; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

*(Note that $f^{(0)} := f$ and $0! := 1$), and the remainder is given by*

$$R_n(f; a, x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

*A simple proof of this theorem can be done by mathematical induction using the integration by parts formula.*

The following corollary concerning the estimation of the remainder is useful when we want to approximate real functions by their Taylor expansions.

**Corollary 7.** *With the above assumptions, we have the estimation*

$$|R_n(f; a, x)| \le \frac{|x-a|^n}{n!} \left| \int_a^x \left| f^{(n+1)}(t) \right| dt \right| \tag{2.2}$$

*or*

$$|R_n(f; a, x)| \le \frac{1}{n!} \frac{|x-a|^{n+\frac{1}{\beta}}}{(n\beta+1)^{\frac{1}{\beta}}} \left| \int_a^x \left| f^{(n+1)}(t) \right|^\alpha dt \right|^{\frac{1}{\alpha}} \tag{2.3}$$

*where $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and the estimation:*

$$|R_n(f; a, x)| \le \frac{|x-a|^{n+1}}{(n+1)!} \max\left\{ \left| f^{(n+1)}(t) \right|,\ t \in [a, x] \text{ or } [x, a] \right\} \tag{2.4}$$

*respectively.*

*Proof.* The inequalities (2.2) and (2.4) are obvious.
Using Hölder's integral inequality, we have that

$$\begin{aligned} \left| \int_a^x (x-t)^n f^{(n+1)}(t)\, dt \right| &\le \left| \int_a^x \left| f^{(n+1)}(t) \right|^\alpha dt \right|^{\frac{1}{\alpha}} \left| \int_a^x |x-t|^{n\beta}\, dt \right|^{\frac{1}{\beta}} \\ &= \left[ \frac{|x-a|^{n\beta+1}}{n\beta+1} \right]^{\frac{1}{\beta}} \left| \int_a^x \left| f^{(n+1)}(t) \right|^\alpha dt \right|^{\frac{1}{\alpha}} \end{aligned}$$

and the inequality (2.3) is also proved. □

The following result for the logarithmic mapping holds.

**Corollary 8.** *Let $a, b > 0$. Then we have the equality:*

$$\ln b - \ln a - \frac{b-a}{a} + \sum_{k=2}^{n} \frac{(-1)^k (b-a)^k}{k a^k} = (-1)^n \int_a^b \frac{(b-t)^n}{t^{n+1}} dt. \tag{2.5}$$

*Proof.* Consider the mapping $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x)=\ln x$. Then

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n},\ n\geq 1,\ x>0,$$

$$T_n(f;a,x)=\ln a+\sum_{k=1}^{n}\frac{(-1)^{k-1}(x-a)^k}{ka^k},\ a>0$$

and

$$R_n(f;a,x)=(-1)^n\int_a^x\frac{(x-t)^n}{t^{n+1}}dt.$$

Now, using $(2.1)$, we have the equality

$$\ln x=\ln a+\sum_{k=1}^{n}\frac{(-1)^{k-1}(x-a)^k}{ka^k}+(-1)^n\int_a^x\frac{(x-t)^n}{t^{n+1}}dt.$$

That is,

$$\ln x-\ln a+\sum_{k=1}^{n}\frac{(-1)^{k}(x-a)^k}{ka^k}=(-1)^n\int_a^x\frac{(x-t)^n}{t^{n+1}}dt,\quad x,a>0.$$

Choosing in the last equality $x=b$, we get $(2.5)$. □

The following inequality for logarithms holds.

**Corollary 9.** *For all $a,b>0$, we have the inequality:*

$$\left|\ln b-\ln a-\frac{b-a}{a}+\sum_{k=2}^{n}\frac{(-1)^k(b-a)^k}{ka^k}\right| \tag{2.6}$$

$$\leq\begin{cases}\frac{|b-a|^n|b^n-a^n|}{na^nb^n}; \\ \frac{|b-a|^{n+\frac{1}{\beta}}}{[(n+1)\alpha-1]^{\frac{1}{\alpha}}(n\beta+1)^{\frac{1}{\beta}}}\left[\frac{\left|b^{(n+1)\alpha-1}-a^{(n+1)\alpha-1}\right|}{b^{(n+1)\alpha-1}a^{(n+1)\alpha-1}}\right]^{\frac{1}{\alpha}}, & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{|b-a|^{n+1}}{n+1}\left[\frac{1}{\min\{a,b\}}\right]^{n+1}. \end{cases}$$

*The equality holds if and only if $a=b$.*

*Proof.* We use Corollary 7 for mapping $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x)=\ln x$ for which we have

$$\int_a^b\left|f^{(n+1)}(t)\right|dt=n!\int_a^b\frac{dt}{t^{n+1}}=n!\left[\left.\frac{t^{-n+1-1}}{-n+1-1}\right|_a^b\right]$$
$$=\frac{n!}{n}\left[\frac{1}{a^n}-\frac{1}{b^n}\right]=\frac{n!}{n}\cdot\frac{b^n-a^n}{a^nb^n}.$$

Also, we have

$$\int_a^b\left|f^{(n+1)}(t)\right|^\alpha dt=(n!)^\alpha\int_a^b\frac{dt}{t^{\alpha(n+1)}}=\frac{(n!)^\alpha}{(n+1)\alpha-1}\cdot\frac{b^{(n+1)\alpha-1}-a^{(n+1)\alpha-1}}{b^{(n+1)\alpha-1}\cdot a^{(n+1)\alpha-1}}$$

and

$$\max\left\{\left|f^{(n+1)}(t)\right|,\ t\in[a,b]\text{ or }t\in[b,a]\right\} = \max\left\{n!\frac{1}{t^{n+1}},\ t\in[a,b]\text{ or }t\in[b,a]\right\} = n!\frac{1}{\min\{a^{n+1},b^{n+1}\}} = n!\left[\frac{1}{\min\{a,b\}}\right]^{n+1}.$$

The equality in (2.6) holds via the representation (2.5) and we omit the details. □

**Remark 1.** *By the concavity property of* $\ln(\cdot)$ *we have*

$$\ln b-\ln a\le\frac{(b-a)}{a}$$

*and then, if we choose* $n=1$ *in* (2.6)*, we get the following counterpart result.*

$$0\le\frac{b-a}{a}-\ln b+\ln a \le\begin{cases}\frac{(b-a)^2}{ab};\\ \frac{|b-a|^{1+\frac{1}{\beta}}\left|b^{2\alpha-1}-a^{2\alpha-1}\right|^{\frac{1}{\alpha}}}{(2\alpha-1)^{\frac{1}{\alpha}}(\beta+1)^{\frac{1}{\beta}}a^{2-\frac{1}{\alpha}}b^{2-\frac{1}{\alpha}}}, & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \frac{(b-a)^2}{2}\cdot\frac{1}{\min^2\{a,b\}}.\end{cases}$$

The equality holds in both inequalities simultaneously if and only if $a=b$.

**Remark 2.** *If we choose* $n=2$ *in* (2.6)*, we get*

$$\left|\ln b-\ln a-\frac{b-a}{a}+\frac{(b-a)^2}{2a^2}\right| \le\begin{cases}\frac{(b-a)^3}{a^2b^2}\cdot\frac{a+b}{2};\\ \frac{|b-a|^{2+\frac{1}{\beta}}\left|b^{3\alpha-1}-a^{3\alpha-1}\right|^{\frac{1}{\alpha}}}{(3\alpha-1)^{\frac{1}{\alpha}}(2\beta+1)^{\frac{1}{\beta}}a^{3-\frac{1}{\alpha}}b^{3-\frac{1}{\alpha}}}, & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \frac{(b-a)^3}{3}\cdot\frac{1}{\min^3\{a,b\}}.\end{cases}$$

*The equality holds in both inequalities simultaneously if and only if* $a=b$.

## 3. Inequalities for Relative Entropy

Let $X$ and $Y$ be two random variables having the probability mass functions $p(x)$, $q(x)$, $x\in\mathfrak{X}$. Then we have the following representation of relative entropy.

**Theorem 5.** *With the above assumptions of* $X$ *and* $Y$*, we have*

$$D(p\|q) = \sum_{x\in\mathfrak{X}}\sum_{k=2}^{n}\frac{(p(x)-q(x))^k}{kp^{k-1}(x)} +(-1)^{n-1}\sum_{x\in\mathfrak{X}}p(x)\int_{p(x)}^{q(x)}\frac{(q(x)-t)^n}{t^{n+1}}dt \tag{3.1}$$

*or*

$$D(p\|q) = \sum_{x\in\mathfrak{X}} p(x) \sum_{k=1}^{n} \frac{(-1)^{k-1}(p(x)-q(x))^k}{kq^k(x)} + (-1)^n \sum_{x\in\mathfrak{X}} p(x) \int_{q(x)}^{p(x)} \frac{(p(x)-t)^n}{t^{n+1}} dt \tag{3.2}$$

*respectively.*

*Proof.* Choose in $(2.5)$ $a = p(x)$, $b = q(x)$, $x \in \mathfrak{X}$ to get

$$\ln q(x) - \ln p(x) - \frac{q(x)-p(x)}{p(x)} + \sum_{k=2}^{n} \frac{(-1)^k (q(x)-p(x))^k}{kp^k(x)} = (-1)^n \int_{p(x)}^{q(x)} \frac{(q(x)-t)^n}{t^{n+1}} dt. \tag{3.3}$$

Multiply $(3.3)$ by $p(x)$ and sum over $x \in \mathfrak{X}$ to get

$$-D(p\|q) - \sum_{x\in\mathfrak{X}} [q(x)-p(x)] + \sum_{x\in\mathfrak{X}} \sum_{k=2}^{n} \frac{(-1)^k (q(x)-p(x))^k}{kp^{k-1}(x)} = (-1)^n \sum_{x\in\mathfrak{X}} p(x) \int_{p(x)}^{q(x)} \frac{(q(x)-t)^n}{t^{n+1}} dt. \tag{3.4}$$

However,

$$\sum_{x\in\mathfrak{X}} [q(x)-p(x)] = 0.$$

Therefore, by $(3.4)$ we get $(3.1)$.
To prove the second equality, choose in $(2.5)$ $b = p(x)$, $a = q(x)$, $x \in \mathfrak{X}$ to get

$$\ln p(x) - \ln q(x) - \frac{p(x)-q(x)}{q(x)} + \sum_{k=2}^{n} \frac{(-1)^k (p(x)-q(x))^k}{kq^k(x)} = (-1)^n \int_{q(x)}^{p(x)} \frac{(p(x)-t)^n}{t^{n+1}} dt. \tag{3.5}$$

Multiply $(3.5)$ by $p(x)$ and sum over $x \in \mathfrak{X}$ to get

$$D(p\|q) = \sum_{x\in\mathfrak{X}} p(x) \sum_{k=1}^{n} \frac{(-1)^{k-1}(p(x)-q(x))^k}{kq^k(x)} + (-1)^n \sum_{x\in\mathfrak{X}} p(x) \int_{q(x)}^{p(x)} \frac{(p(x)-t)^n}{t^{n+1}} dt$$

from where we get $(3.2)$. □

Using Corollary 9, we can give the following result containing an approximation of the relative entropy.

**Theorem 6.** *With the above assumption over $X$ and $Y$, we have*

$$\left| D(p\|q) - \sum_{x\in\mathfrak{X}} \sum_{k=2}^{n} \frac{(p(x)-q(x))^k}{kp^{k-1}(x)} \right|$$

$$\leq \quad M := \begin{cases} \frac{1}{n}\sum_{x\in\mathfrak{X}} \frac{|q(x)-p(x)|^n |q^n(x)-p^n(x)|}{p^{n-1}(x)q^n(x)}; \\ \frac{1}{[(n+1)\alpha-1]^{\frac{1}{\alpha}}(n\beta+1)^{\frac{1}{\beta}}} \sum_{x\in\mathfrak{X}} p(x)|q(x)-p(x)|^{n+\frac{1}{\beta}} \\ \quad \times \left[ \frac{\left|q^{(n+1)\alpha-1}(x)-p^{(n+1)\alpha-1}(x)\right|}{q^{(n+1)\alpha-1}(x)p^{(n+1)\alpha-1}(x)} \right]^{\frac{1}{\alpha}}; \\ \frac{1}{n+1}\sum_{x\in\mathfrak{X}} p(x)|q(x)-p(x)|^{n+1} \\ \quad \times \left[ \frac{1}{\min\{p(x),q(x)\}} \right]^{n+1}; \end{cases}$$

*and*

$$\left| D(p\|q) - \sum_{x\in\mathfrak{X}} p(x) \sum_{k=1}^{n} \frac{(-1)^{k-1}(p(x)-q(x))^k}{kq^k(x)} \right| \leq M, \tag{3.6}$$

*respectively. The equality holds in both inequalities if and only if $p(x) = q(x)$, $x \in \mathfrak{X}$.*

*Proof.* Proof for the first inequality is obvious by Corollary 9, choosing $a = p(x)$, $b = q(x)$, multiplying by $q(x)$ and sum over $x \in \mathfrak{X}$. Proof for second inequality is obvious by Corollary 9, choosing $b = p(x)$, $a = q(x)$, multiplying by $q(x)$ and sum over $x \in \mathfrak{X}$. □

**Corollary 10.** *Under the assumptions from Theorem 6 for $n = 1$, we have*

$$D(p\|q) \leq M_1 \tag{3.7}$$

*where*

$$M_1 := \begin{cases} \sum_{x\in\mathfrak{X}} \frac{(q(x)-p(x))^2}{q(x)}; \\ \frac{1}{(2\alpha-1)^{\frac{1}{\alpha}}(\beta+1)^{\frac{1}{\beta}}} \sum_{x\in\mathfrak{X}} p(x)|q(x)-p(x)|^{1+\frac{1}{\beta}} \times \frac{\left|q^{2\alpha-1}(x)-p^{2\alpha-1}(x)\right|^{\frac{1}{\alpha}}}{q^{2-\frac{1}{\alpha}}(x)p^{2-\frac{1}{\alpha}}(x)}; \\ \frac{1}{2}\sum_{x\in\mathfrak{X}} p(x)(q(x)-p(x))^2 \times \frac{1}{\min^2\{p(x),q(x)\}}; \end{cases}$$

*and*

$$0 \leq \sum_{x\in\mathfrak{X}} \frac{p(x)}{q(x)}(p(x)-q(x)) - D(p\|q) \leq M_1 \tag{3.8}$$

*respectively, where $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.*

**Remark 3.** *The first inequality in (3.7) is equivalent to (see also [2]):*

$$D(p\|q) \leq \sum_{x\in\mathfrak{X}} \frac{p^2(x)}{q(x)} - 1$$

*with equality if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.*

*We introduce notation $M_{12}$ for the summation part of the second term in $M_1$ and apply Hölder's discrete inequality. Then we can write*

$$M_{12} := \sum_{x\in\mathfrak{X}} p(x)\left|q(x)-p(x)\right|^{\frac{\beta+1}{\beta}} \times \frac{\left|q^{2\alpha-1}(x)-p^{2\alpha-1}(x)\right|^{\frac{1}{\alpha}}}{q^{2-\frac{1}{\alpha}}(x)\,p^{2-\frac{1}{\alpha}}(x)}$$

$$\leq \left(\sum_{x\in\mathfrak{X}} p(x)\left|q(x)-p(x)\right|^{\beta+1}\right)^{\frac{1}{\beta}}\left(\sum_{x\in\mathfrak{X}}\frac{\left|q^{2\alpha-1}(x)-p^{2\alpha-1}(x)\right|}{q^{2\alpha-1}(x)\,p^{2\alpha-2}(x)}\right)^{\frac{1}{\alpha}} := \tilde{M}_{12}$$

*and then from second inequality of (3.7) we get the inequality*

$$D(p\|q) \leq \frac{1}{(2\alpha-1)^{\frac{1}{\alpha}}(\beta+1)^{\frac{1}{\beta}}}\tilde{M}_{12}.$$

*For $\alpha=\beta=2$, we get the particular inequality*

$$D(p\,\|q) \leq \frac{1}{3}\left(\sum_{x\in\mathfrak{X}} p(x)\left|q(x)-p(x)\right|^{3}\right)^{\frac{1}{2}} \times \left(\sum_{x\in\mathfrak{X}}\frac{\left|q^{3}(x)-p^{3}(x)\right|}{q^{3}(x)\,p^{2}(x)}\right)^{\frac{1}{2}}.$$

*If we assume that*

$$\min_{x\in\mathfrak{X}}\left[\min\left\{p(x),q(x)\right\}\right] = \delta > 0 \tag{3.9}$$

*then from the third inequality of (3.7) we have the inequality*

$$D(p\|q) \leq \frac{1}{2\delta^2}\sum_{x\in\mathfrak{X}} p(x)\left(q(x)-p(x)\right)^2.$$

**Remark 4.** *Since*

$$\sum_{x\in\mathfrak{X}}\frac{p(x)}{q(x)}\left(p(x)-q(x)\right) = \sum_{x\in\mathfrak{X}}\frac{p^2(x)}{q(x)} - 1,$$

*then the first inequality in (3.8) is obvious.*
*Using Hölder's inequality, from second inequality in (3.8) we get*

$$\sum_{x\in\mathfrak{X}}\frac{p^2(x)}{q(x)} - 1 - D(p\|q) \leq \frac{1}{(2\alpha-1)^{\frac{1}{\alpha}}(\beta+1)^{\frac{1}{\beta}}}\tilde{M}_{12}.$$

*If, as above, we assume that (3.9) holds, then from third inequality in (3.8) we have*

$$\sum_{x\in\mathfrak{X}}\frac{p^2(x)}{q(x)} - 1 - D(p\|q) \leq \frac{1}{2\delta^2}\sum_{x\in\mathfrak{X}} p(x)\left(q(x)-p(x)\right)^2.$$

## 4. **Inequalities for the Entropy Mapping**

Let $X$ be a random variable having the probability mass function $p(x)$, $x\in\mathfrak{X}$. Consider the *entropy mapping*

$$H(X) = \sum_{x\in\mathfrak{X}} p(x)\ln\frac{1}{p(x)}.$$

We have the following representation of $H(X)$.

**Theorem 7.** *With the above assumption for $X$, we have*

$$H(X) = \ln|\mathfrak{X}| - \sum_{x\in\mathfrak{X}}\sum_{k=2}^{n}\frac{(|\mathfrak{X}|\,p(x)-1)^k}{k\,|\mathfrak{X}|^k\,p^{k-1}(x)} + \frac{(-1)^n}{|\mathfrak{X}|^n}\sum_{x\in\mathfrak{X}}p(x)\int_{p(x)}^{\frac{1}{|\mathfrak{X}|}}\frac{(1-|\mathfrak{X}|\,t)^n}{t^{n+1}}dt \tag{4.1}$$

*or*

$$H(X) = \ln|\mathfrak{X}| + \sum_{x\in\mathfrak{X}}p(x)\sum_{k=1}^{n}\frac{(-1)^k(|\mathfrak{X}|\,p(x)-1)^k}{k} + (-1)^{n+1}\sum_{x\in\mathfrak{X}}p(x)\int_{\frac{1}{|\mathfrak{X}|}}^{p(x)}\frac{(p(x)-t)^n}{t^{n+1}}dt \tag{4.2}$$

*respectively.*

*Proof.* Put in (3.1) $q=u$, where $u$ is the uniform distribution on $\mathfrak{X}$, i.e., $u(x)=\frac{1}{|\mathfrak{X}|}$, $|\mathfrak{X}|$ is the number of elements in $\mathfrak{X}$. Then

$$\begin{aligned}
&\ln|\mathfrak{X}| - H(X)\\
&= \sum_{x\in\mathfrak{X}}\sum_{k=2}^{n}\frac{\left(p(x)-\frac{1}{|\mathfrak{X}|}\right)^k}{kp^{k-1}(x)} + (-1)^{n-1}\sum_{x\in\mathfrak{X}}p(x)\int_{p(x)}^{\frac{1}{|\mathfrak{X}|}}\frac{\left(\frac{1}{|\mathfrak{X}|}-t\right)^n}{t^{n+1}}dt\\
&= \sum_{x\in\mathfrak{X}}\sum_{k=2}^{n}\frac{(|\mathfrak{X}|\,p(x)-1)^k}{k\,|\mathfrak{X}|^k\,p^{k-1}(x)} + \frac{(-1)^{n-1}}{|\mathfrak{X}|^n}\sum_{x\in\mathfrak{X}}p(x)\int_{p(x)}^{\frac{1}{|\mathfrak{X}|}}\frac{(1-|\mathfrak{X}|\,t)^n}{t^{n+1}}dt
\end{aligned}$$

from where results (4.1).
Put in (3.2) $q=u$, to get

$$\begin{aligned}
&\ln|\mathfrak{X}| - H(X)\\
&= \sum_{x\in\mathfrak{X}}p(x)\sum_{k=1}^{n}\frac{(-1)^{k-1}\left(p(x)-\frac{1}{|\mathfrak{X}|}\right)^k}{k\frac{1}{|\mathfrak{X}|^k}} + (-1)^n\sum_{x\in\mathfrak{X}}p(x)\int_{\frac{1}{|\mathfrak{X}|}}^{p(x)}\frac{(p(x)-t)^n}{t^{n+1}}dt\\
&= \sum_{x\in\mathfrak{X}}p(x)\sum_{k=1}^{n}\frac{(-1)^{k-1}(|\mathfrak{X}|\,p(x)-1)^k}{k} + (-1)^n\sum_{x\in\mathfrak{X}}p(x)\int_{\frac{1}{|\mathfrak{X}|}}^{p(x)}\frac{(p(x)-t)^n}{t^{n+1}}dt,
\end{aligned}$$

from where results (4.2). □

Using Theorem 6, we can state the following result concerning the approximation of the entropy mapping.

**Theorem 8.** *With the above assumption for $X$, we have*

$$\left| H\left(X\right) - \ln\left|\mathfrak{X}\right| + \sum_{x\in\mathfrak{X}}\sum_{k=2}^{n}\frac{\left(\left|\mathfrak{X}\right| p\left(x\right)-1\right)^{k}}{k\left|\mathfrak{X}\right|^{k} p^{k-1}\left(x\right)}\right|$$

$$\leq \quad \mu := \begin{cases} \frac{1}{n\left|\mathfrak{X}\right|^{n}}\sum_{x\in\mathfrak{X}}\frac{\left|1-\left|\mathfrak{X}\right|p(x)\right|^{n}\left|1-\left|\mathfrak{X}\right|^{n}p^{n}(x)\right|}{p^{n-1}(x)}; \\ \frac{1}{\left[(n+1)\alpha-1\right]^{\frac{1}{\alpha}}(n\beta+1)^{\frac{1}{\beta}}}\sum_{x\in\mathfrak{X}}p\left(x\right)\left|\frac{1}{\left|\mathfrak{X}\right|}-p\left(x\right)\right|^{n+\frac{1}{\beta}}\times\left[\frac{\left|1-\left|\mathfrak{X}\right|^{(n+1)\alpha-1}p^{(n+1)\alpha-1}(x)\right|}{p^{(n+1)\alpha-1}(x)}\right]^{\frac{1}{\alpha}}; \\ \frac{1}{n+1}\sum_{x\in\mathfrak{X}}p\left(x\right)\left|\frac{1}{\left|\mathfrak{X}\right|}-p\left(x\right)\right|^{n+1}\times\left[\frac{1}{\min\left\{p(x),\frac{1}{\left|\mathfrak{X}\right|}\right\}}\right]^{n+1}; \end{cases}$$

*and*

$$\left| H\left(X\right) - \ln\left|\mathfrak{X}\right| - \sum_{x\in\mathfrak{X}} p\left(x\right)\sum_{k=1}^{n}\frac{\left(-1\right)^{k}\left(\left|\mathfrak{X}\right| p\left(x\right)-1\right)^{k}}{k}\right| \leq \mu.$$

## 5. **Inequalities for Mutual Information**

Let $X$ and $Y$ be random variables having the probability mass functions $p\left(x\right)$, $q\left(y\right)$, $x\in\mathfrak{X}$, $y\in\mathcal{Y}$. Consider the *mutual information* [1]

$$I\left(X,Y\right) = \sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}} p\left(x,y\right)\ln\frac{p\left(x,y\right)}{p\left(x\right)q\left(y\right)}.$$

We have the following representation for $I\left(X,Y\right)$.

**Theorem 9.** *With the above assumption for $X$ and $Y$, we get*

$$\begin{aligned} I\left(X,Y\right) &= \sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}}\sum_{k=2}^{n}\frac{\left(p\left(x,y\right)-p\left(x\right)q\left(y\right)\right)^{k}}{kp^{k-1}\left(x,y\right)} \\ &\quad +\left(-1\right)^{n-1}\sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}}p\left(x,y\right)\times\int_{p(x,y)}^{p(x)q(y)}\frac{\left(p\left(x\right)q\left(y\right)-t\right)^{n}}{t^{n+1}}dt \end{aligned}$$

*or*

$$\begin{aligned} I\left(X,Y\right) &= \sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}}p\left(x,y\right)\sum_{k=1}^{n}\frac{\left(-1\right)^{k-1}\left(p\left(x,y\right)-p\left(x\right)q\left(y\right)\right)^{k}}{kp^{k}\left(x\right)q^{k}\left(y\right)} \\ &\quad +\left(-1\right)^{n}\sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}}p\left(x,y\right)\int_{p(x)q(y)}^{p(x,y)}\frac{\left(p\left(x,y\right)-t\right)^{n}}{t^{n+1}}dt. \end{aligned}$$

*The proof follows by Theorem 5, taking into account that*

$$I\left(X,Y\right) = D\left(p\left(x,y\right)\|p\left(x\right)q\left(y\right)\right).$$

*Finally, using Theorem 6, we can state the following estimation of the mutual information.*

**Theorem 10.** *With the above assumption over $X$ and $Y$, we have:*

$$\left| I(X,Y) - \sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}}\sum_{k=2}^{n}\frac{(p(x,y)-p(x)q(y))^k}{kp^{k-1}(x,y)}\right|$$

$$\leq \tilde{M} := \begin{cases} \frac{1}{n}\sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}}\frac{|p(x)q(y)-p(x,y)|^n}{p^{n-1}(x,y)p^{n-1}(x)q^{n-1}(y)} \times |p^n(x)q^n(y)-p^n(x,y)|; \\ \frac{1}{[(n+1)\alpha-1]^{\frac{1}{\alpha}}(n\beta+1)^{\frac{1}{\beta}}}\sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}} p(x,y)\times|p(x)q(y)-p(x,y)|^{n+\frac{1}{\beta}} \\ \qquad \times\left[\frac{\left|p^{(n+1)\alpha-1}(x)q^{(n+1)\alpha-1}(y)-p^{(n+1)\alpha-1}(x,y)\right|}{p^{(n+1)\alpha-1}(x)q^{(n+1)\alpha-1}(y)p^{(n+1)\alpha-1}(x,y)}\right]^{\frac{1}{\alpha}}; \\ \frac{1}{n+1}\sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}} p(x,y)\times|p(x)q(y)-p(x,y)|^{n+1} \\ \qquad \times\left[\frac{1}{\min\{p(x,y),p(x)q(y)\}}\right]^{n+1}; \end{cases}$$

*and*

$$\left| I(X,Y) - \sum_{x\in\mathfrak{X}}\sum_{y\in\mathcal{Y}} p(x,y)\sum_{k=1}^{n}\frac{(-1)^{k-1}(p(x,y)-p(x)q(y))^k}{kp^k(x)q^k(y)}\right| \leq \tilde{M}.$$

*The equality holds in both inequalities simultaneously if and only if $X$ and $Y$ are independent.*

For other results related to the entropy mapping and the mutual information, we recommend the recent papers [2]-[8] where further references are given.

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School of Communications and Informatics,, Victoria University, PO Box 14428, MCMC, Melbourne, Victoria 8001

*E-mail address*: sever@matilda.vu.edu.au

Royal Australian Air Force, ARDU, PoBox 1500, Salisbury SA 5108,

*E-mail address*: vgluscev@spam.adelaide.edu.au