AN EXTENSION OF THE GARRETT-STANOJEVIC CLASS

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Abstract. In this paper, some corrections of the paper published by the author of the present note in **Approximation theory & Applications, Vol. 16, N 1, 46-51** are made. The extension is made for the theorem of [6], by considering the classes $\mathcal{S}_{pr}$ and $\mathcal{C}_r$, $1 < p \le 2$, $r \in \{0, 1, 2, \ldots\}$ instead of $\mathcal{S}_p$ and $\mathcal{C}$. Namely, it is shown that the class $\mathcal{S}_{pr}$ is a subclass of $\mathcal{C}_r \cap \mathcal{BV}$, $1 < p \le 2$, $r \in \{0, 1, 2, \ldots\}$, where $\mathcal{BV}$ is the class of null sequences of bounded variation, and $\mathcal{C}_r$, $r \in \{0, 1, 2, \ldots\}$ is the extension of the Garrett-Stanojević class.

1. Introduction

Several authors have studied the problem of L^1-convergence of the cosine series (C)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \,.$$

Let $\mathcal{F}$ be the class of sequences of Fourier coefficients of even and 2π-periodic function $f \in L^1(0, \pi)$.

It's well-known that, in general it doesn,t follow from $\{a_n\} \in \mathcal{F}$ that

$$\|S_n - f\| = o(1)\,, \quad n \to \infty\,, \qquad (*)$$

where S_n are the partial sums of (C).

There are examples of subclasses of F for which

$$a_n \log n = o(1)\,, \quad n \to \infty$$

is necessary and sufficient condition for $(*)$.

Telyakovskii [5] introduced the following classical example of class of L^1-convergence of Fourier series.

A null sequence $\{a_k\}$ belongs to $\mathcal{S}$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \le A_k$, for all k.

Let $\mathcal{BV}$ denote the class of null-sequences of bounded variation. Garrett and Stanojević [1] introduced the following class $\mathcal{C}$. A null sequence $\{a_k\}$ belongs to the class $\mathcal{C}$ if for every $\varepsilon > 0$ there exists $\delta > 0$, independet of n, and such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon\,,$$

for all n, where $D_n(x)$ is the Dirichlet kernel.

As a corollary to their main result in [1], they proved that $\mathcal{C} \cap \mathcal{BV}$ is a class of L^1-convergence.

Later, Garrett, Rees and Stanojević [2] proved that $\mathcal{S} \subset \mathcal{BV} \cap \mathcal{C}$.

In [4] Č. V. Stanojević and V. B. Stanojević extended the class $\mathcal{S}$ of Telyakovskii.

Namely, they defined a class $\mathcal{S}_p$, $p > 1$ as follows: a sequence $\{a_k\}$ belongs to the class $\mathcal{S}_p$, $p > 1$ or $\{a_k\} \in \mathcal{S}_p$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p} = O(1)\,.$$

In [6] we proved that $\mathcal{S}_p$ is a subclass of the class $\mathcal{C} \cap \mathcal{BV}$.

Now, we define a class $\mathcal{S}_{pr}$, $1 < p \le 2$, $r \in \{0, 1, 2, \ldots\}$ as follows: $\{a_k\} \in \mathcal{S}_{pr}$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ sush that $\sum_{k=1}^{\infty} k^r A_k < \infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p} = O(1)\,.$$

When $r = 0$, denote $\mathcal{S}_p = \mathcal{S}_{pr}$.

We shall call a null sequence $\{a_k\}$ belongs to the class $\mathcal{C}_r$, i.e. $\{a_k\} \in \mathcal{C}_r$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k^{(r)}(x)\right| dx < \varepsilon\,,$$

for all n, where $D_k^{(r)}(x)$ is the r-th derivate of the Dirichlet kernel.

The class $\mathcal{C}_r$, we shall call an extension of the Garrett-Stanojević class. In this paper we shall prove that $\mathcal{S}_{pr}$, is a subclass of $\mathcal{BV} \cap \mathcal{C}_r$.

2. Lemmas

For the proof of our theorem we need the following lemmas:

Lemma 1. [3] *Let r be a nonnegative integer, and $x \in (0, \pi]$. Then*

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1}\frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}}\varphi_k(x) +$$

$$+\frac{\left(n+\frac{1}{2}\right)^r \sin\left[\left(n+\frac{1}{2}\right)x + \frac{r\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)}\,,$$

where the same φ_k denotes various analytical function of x independent of n.

Lemma 2. *Let $\{\alpha_j\}_{j=1}^{k}$ be a sequence of real numbers.*

Then the following relation holds for $v = 0, 1, 2, \ldots, r$ and $r = 0, 1, 2, \ldots$

$$T_k = \int\limits_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx = O_p\left[k\left(k^{pr-1}\sum_{j=1}^{k}|\alpha_j|^p\right)^{1/p}\right],$$

where O_p depends only on p.

Proof. Applying first Hölder inequality, yields:

$$T_k = \int\limits_{\pi/k}^{\pi} \frac{1}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \left| \sum_{j=1}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right| dx$$

$$\le \left[\int\limits_{\pi/k}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{(r+1-v)p}}\right]^{1/p} \left\{\int\limits_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx\right\}^{1/q}.$$

Since

$$\int\limits_{\pi/k}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{(r+1-v)p}} \le \pi^{(r+1-v)p} \int\limits_{\pi/k}^{\pi} \frac{dx}{x^{(r+1-v)p}} \le \frac{\pi k^{(r+1-v)p-1}}{(r+1-v)p-1} \le \frac{\pi k^{(r+1-v)p-1}}{p-1},$$

we have:

$$T_k \le \left(\frac{\pi}{p-1}\right)^{\frac{1}{p}} \left(k^{(r+1-v)p-1}\right)^{\frac{1}{p}} \left\{\int\limits_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx\right\}^{\frac{1}{q}}.$$

Then using the Hausdorff-Young inequality we get:

$$\left\{\int\limits_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right] \right|^q dx\right\}^{1/q} = O\left[\left(\sum_{j=1}^{k}|\alpha_j|^p j^{vp}\right)^{1/p}\right].$$

Finally,

$$T_k = O_p\left[\left(k^{(r+1-v)p-1}\right)^{1/p}\left(\sum_{j=1}^{k}|\alpha_j|^p j^{vp}\right)^{1/p}\right] = O_p\left[\left(k^{(r+1)p-1}\right)^{1/p}\left(\sum_{j=1}^{k}|\alpha_j|^p\right)^{1/p}\right]$$

$$= O_p\left[k\left(k^{pr-1}\sum_{j=1}^{k}|\alpha_j|^p\right)^{1/p}\right],$$

where O_p depends only on p.

2. Main result

Theorem. For each $1 < p \leq 2$ and $r = 0, 1, 2, \ldots$ the following embedding relation holds $\mathcal{S}_{pr} \subset \mathcal{C}_r \cap \mathcal{BV}$.

Proof. We have:

$$\sum_{k=1}^{n} |\Delta a_k| \leq \sum_{k=1}^{n} k^r |\Delta a_k| = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} j^r + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} j^r$$

$$\leq \sum_{k=1}^{n-1} (\Delta A_k) k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + n^{r+1} A_n \left(\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p}$$

$$= O(1) \left(\sum_{k=1}^{n-1} (\Delta A_k) k^{r+1} + n^{r+1} A_n \right)$$

$$= O\left(\sum_{k=1}^{n} k^r A_k \right)$$

Letting $n \to \infty$, we get $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, i.e. $\{a_n\} \in \mathcal{BV}$.

Then, applying Abel's transformation, we have:

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq$$

$$\leq \sum_{k=n}^{\infty} (\Delta A_k) \int_0^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx + A_n \int_0^{\pi} \left| \sum_{j=1}^{n-1} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx\,.$$

Now we estimate the integral

$$\int_0^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_k^{(r)}(x) \right| dx = \int_0^{\pi/k} + \int_{\pi/k}^{\pi} = I_k + J_k\,.$$

Applying the inequality $D_n^{(r)}(x) = O(n^{r+1})$, we have:

$$I_k \leq \alpha \sum_{j=1}^{k} j^r \frac{|\Delta a_j|}{A_j} \leq \alpha k^r \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \leq \alpha k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} = O(k^{r+1})$$

where α is an absolute constant.

Applying the Lemma 1, let us estimate the second integral:

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx \leq$$

$$\leq \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \left(\sum_{v=0}^{r-1} \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v(x) \right) \right| dx +$$

$$+ \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{\left(j+\frac{1}{2}\right)^r \sin\left[\left(j+\frac{1}{2}\right)x + \frac{r\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)} \right| dx$$

$$= \lambda_k + \mu_k \,.$$

Since φ_v are bounded, we have:

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v \right| dx \leq$$

$$\leq B \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \right| dx$$

where B is a positive constant, and $\alpha_j = \dfrac{\Delta a_j}{A_j}$, $j = 1, 2, \ldots, k$.

Applying Lemma 2 to the last integral, we get:

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right)x + \frac{v\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-v}} \varphi_v(x) \right| dx =$$

$$= O_p\left[k \left(k^{pr-1} \sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \right] = O_p\left(k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} \right)$$

Since r is a finite value, we have $\lambda_k = O_p(k^{r+1})$.

Similarly, we can get: $\mu_k = O_p(k^{r+1})$.

Hence

$$\int_{0}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O(k^{r+1}) + O_p(k^{r+1}) = O_p(k^{r+1})\,.$$

Finally,

$$\int_{0}^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx \leq O_p(1) \left[\sum_{k=n}^{\infty} k^{r+1} (\Delta A_k) + n^{r+1} A_n \right]$$

$$= O_p(1)\left[\lim_{N\to\infty}\sum_{k=n}^{N-1} k^{r+1}(\Delta A_k) + n^{r+1}A_n\right]$$

$$= O_p\left(\sum_{k=n}^{\infty} k^r A_k\right) + o(1) = o(1)\,, \quad n\to\infty\,,$$

since

$$N^{r+1}A_N = o(1)\,, \quad N\to\infty\,.$$

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