A SIMPLE PROOF OF THE ENTROPY INEQUALITY
$\ln p \cdot \ln q \leq H(p,q) \leq \ln p \cdot \ln q / \ln 2$

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1. Introduction

The purpose of this note is to give a simple proof to the following entropy inequality:

$$\ln p \cdot \ln q \leq H(p,q) \leq \ln p \cdot \ln q / \ln 2$$

where p and q are positive real numbers with $p+q=1$ and $H(p,q) = -p\ln p - q \ln q$ is the entropy of the probability vector (p,q). This problem was suggested by F.Tops∅e in an e-mail to the RGMIA, and communicated to us by our teacher, B.Djafari Rouhani. We refer to [1] for more information on this inequality.

2. The Results

In order to prove the inequality, we first state the following simple lemma.

Lemma 1. *The function* $f : [0,1] \to R$ *defined by* $f(0) = 0, f(1) = 1$ *and* $f(x) = (x-1)/\ln x$ *for* $0 < x < 1$, *is concave on* $[0,1]$.

Proof. Since f is continuous on $[0,1]$ and infinitely differentiable on $(0,1)$, it suffices to show that $f'' \leq 0$. A simple computation gives $f''(x) = -g(x)/x^2(\ln x)^3$ where $g(x) = (x+1)\ln x + 2(1-x)$. Thus, it is enough to show that $g(x) \leq 0$. But we have $\lim_{x\to 1^-} g(x) = 0$ and $g'(x) = 1/x - 1 - \ln 1/x > 0$ for $0 < x < 1$ since $\ln y \leq y - 1$ for $y \geq 1$. This shows that $g(x) \leq 0$ and completes the proof of the lemma. ∎

Now we prove the inequality.

Theorem 1. *Let* p *and* q *be positive real numbers with* $p+q=1$. *Then the following inequality holds:* $\ln p . \ln q \leq -p\ln p - q\ln q \leq \ln p . \ln q / \ln 2$

Proof. Without loss of generality we may assume that $0 < p \leq 1/2 \leq q < 1$. The inequality to prove is equivalent to : $1 \leq (q-1)/\ln q + (p-1)/\ln p \leq 1/\ln 2$; using the function f introduced in the lemma, this is equivalent to : $f(0) + f(1) \leq f(p) + f(q) \leq 2f(1/2)$, i.e., equivalent to: $(f(1) - f(q))/(1-q) \leq (f(p) - f(0))/(p-0)$ and $(f(q) - f(1/2))/(q - 1/2) \leq (f(1/2) - f(p))/(1/2 - p)$ which follow both from the concavity of f, and completes the proof of the theorem. ∎

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References

[1] F.Tops∅e, Bounds for entropy and divergence for distributions over a two-element set, preprint, personal communication.

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