A TRAPEZOID TYPE INEQUALITY FOR DOUBLE INTEGRALS

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ABSTRACT. In this paper, we point out a trapezoid like inequality for double integrals and apply it in connection with the Grüss inequality.

1. INTRODUCTION

In the recent papers [1] and [2], the authors proved the following inequality of the Ostrowski type for double integrals.

Theorem 1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$. Then we have the inequality:*

$$\begin{aligned}(1.1)\quad & \Bigg| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt \\ & -\frac{1}{d-c}\int_c^d f(x,s)\,ds - \frac{1}{b-a}\int_a^b f(t,y)\,dt + f(x,y) \Bigg| \\ \leq & \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4} + \left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right](b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_\infty \\ \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_\infty([a,b]\times[c,d]); \\ \frac{1}{(q+1)^{\frac{2}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left[\left(\frac{y-c}{d-c}\right)^{q+1} + \left(\frac{d-y}{d-c}\right)^{q+1}\right]^{\frac{1}{q}} \\ \times [(b-a)(d-c)]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_p \\ \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2} + \left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_1 \\ \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_1([a,b]\times[c,d]) \end{cases}\end{aligned}$$

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for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$\begin{aligned}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{\infty} &:= \sup_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|,\\ \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p} &:= \left(\int_a^b\int_c^d\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|^p dsdt\right)^{\frac{1}{p}}\end{aligned}$$

if $p\in[1,\infty)$.

The best inequality we can get from (1.1) is the one for which $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, obtaining:

Corollary 1. *With the assumptions in Theorem 1, we have the following mid-point type inequality:*

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds\right.\\&\left.-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\&\le\begin{cases}\frac{1}{16}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{\infty}(b-a)(d-c) & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_{\infty}([a,b]\times[c,d]);\\ \frac{1}{4(q+1)^{\frac{2}{q}}}\left[(b-a)(d-c)\right]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p} & \\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_p([a,b]\times[c,d]), \quad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{1} & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1([a,b]\times[c,d]).\end{cases}\end{aligned}\tag{1.2}$$

For some applications of the above results in Numerical Integration for cubature formulae see [1] and [2].

Another result of Ostrowski type was proved in [4].

Theorem 2. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a mapping as in Theorem 1. Then we have the inequality:*

$$\begin{aligned}&\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|\\&\le M_1(x)+M_2(y)+M_3(x,y),\end{aligned}\tag{1.3}$$

where

$$M_1(x)=\begin{cases}\frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]}{b-a}\left\|\frac{\partial f}{\partial t}\right\|_{\infty}, & \text{if } \frac{\partial f}{\partial t}\in L_{\infty}([a,b]\times[c,d]);\\ \frac{\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}}{(b-a)[(d-c)]^{\frac{1}{p_1}}}\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f}{\partial t}\in L_{p_1}([a,b]\times[c,d]),\\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1;\\ \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_{1}, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d]).\end{cases}$$

$$M_2(y)=\begin{cases}\frac{\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{d-c}\left\|\frac{\partial f}{\partial s}\right\|_{\infty}, & \text{if } \frac{\partial f}{\partial s}\in L_{\infty}([a,b]\times[c,d]);\\ \frac{\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}}{[(b-a)]^{\frac{1}{p_2}}(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \frac{\partial f}{\partial s}\in L_{p_2}([a,b]\times[c,d]);\\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1;\\ \frac{\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_{1}, & \text{if } \frac{\partial f}{\partial s}\in L_1([a,b]\times[c,d]);\end{cases}$$

and

$$M_3(x,y)=\begin{cases}\frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{\infty},\\ \text{if } \frac{\partial^2 f}{\partial s\partial t}\in L_{\infty}([a,b]\times[c,d]);\\ \frac{\left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3},\\ \text{if } \frac{\partial^2 f}{\partial s\partial t}\in L_{p_3}([a,b]\times[c,d]),\ p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1;\\ \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{1},\\ \text{if } \frac{\partial^2 f}{\partial s\partial t}\in L_1([a,b]\times[c,d]);\end{cases}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|\cdot\|_p$ $(1\le p\le\infty)$ *are the usual* $p-$*norms on* $[a,b]\times[c,d]$.

Corollary 2. *With the assumptions in Theorem 1, we have the inequality*

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \tag{1.4}$$
$$\le \tilde{M}_1+\tilde{M}_2+\tilde{M}_3,$$

where

$$\tilde{M}_1:=\begin{cases}\frac{1}{4}(b-a)\left\|\frac{\partial f}{\partial t}\right\|_{\infty}, & \text{if } \frac{\partial f}{\partial t}\in L_{\infty}([a,b]\times[c,d])\\ \frac{1}{2}\left[\frac{(b-a)^{\frac{1}{q_1}}}{(q_1+1)^{\frac{1}{q_1}}(d-c)^{\frac{1}{p_1}}}\right]\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f}{\partial t}\in L_{p_1}([a,b]\times[c,d])\\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1;\\ \frac{1}{2(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_{1}, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d])\end{cases}$$

$$\tilde{M}_2:=\begin{cases}\frac{1}{4}(d-c)\left\|\frac{\partial f}{\partial s}\right\|_{\infty}, & \text{if } \frac{\partial f}{\partial t}\in L_{\infty}([a,b]\times[c,d])\\ \frac{1}{2}\left[\frac{(d-c)^{\frac{1}{q_2}}}{(q_2+1)^{\frac{1}{q_2}}(b-a)^{\frac{1}{p_2}}}\right]\left\|\frac{\partial f}{\partial s}\right\|_{p_2} & \text{if } \frac{\partial f}{\partial t}\in L_{p_2}([a,b]\times[c,d])\\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1;\\ \frac{1}{2(b-a)}\left\|\frac{\partial f}{\partial s}\right\|_{1}, & \text{if } \frac{\partial f}{\partial t}\in L_1([a,b]\times[c,d])\end{cases}$$

and

$$\tilde{M}_3 := \begin{cases} \frac{1}{16}(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_{\infty}, & \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_{\infty}([a,b]\times[c,d]); \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q_3}}(d-c)^{\frac{1}{q_3}}}{(q_3+1)^{\frac{2}{q_3}}}\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_{p_3}, & \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_{p_3}([a,b]\times[c,d]); \\ & p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1, \\ \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_{1}, & \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_1([a,b]\times[c,d]). \end{cases}$$

2. Some Integral Equalities

Let us start with the following integral identity.

Theorem 3. *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a continuous mapping on $[a,b]\times[c,d]$ such that $\frac{\partial f(a,\cdot)}{\partial y}$, $\frac{\partial f(b,\cdot)}{\partial y}$ are continuous on $[c,d]$, $\frac{\partial f(\cdot,c)}{\partial x}$, $\frac{\partial f(\cdot,d)}{\partial x}$ are continuous on $[a,b]$ and $\frac{\partial^2 f(\cdot,\cdot)}{\partial x \partial y}$ is continuous on $[a,b]\times[c,d]$. Then we have the identity:*

$$\begin{aligned} &\int_a^b\int_c^d f(x,y)\,dydx + (b-a)\int_c^d\left(y-\frac{c+d}{2}\right)\left[\frac{\frac{\partial f(a,y)}{\partial y}+\frac{\partial f(b,y)}{\partial y}}{2}\right]dy \\ &+(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\frac{\partial f(x,c)}{\partial x}+\frac{\partial f(x,d)}{\partial x}}{2}\right]dx \\ =\ &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c) \\ &+\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx. \end{aligned} \tag{2.1}$$

Proof. A simple integration by parts gives

$$\int_\alpha^\beta h(x)\,dx = \frac{h(\alpha)+h(\beta)}{2}(\beta-\alpha) - \int_\alpha^\beta\left(x-\frac{\alpha+\beta}{2}\right)h'(x)\,dx, \tag{2.2}$$

provided that $h : [\alpha,\beta] \to \mathbb{R}$ is absolutely continuous on $[\alpha,\beta]$.

Using (2.2), we can write:

$$\int_a^b f(x,y)\,dx = (b-a)\frac{f(a,y)+f(b,y)}{2} - \int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,y)}{\partial x}dx \tag{2.3}$$

for all $y \in [c,d]$.

Integrating (2.3) on the interval $[c,d]$, we obtain

$$\begin{aligned} \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy &= \frac{1}{2}(b-a)\left[\int_c^d f(a,y)\,dy + \int_c^d f(b,y)\,dy\right] \\ &\quad -\int_c^d\left(\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,y)}{\partial x}dx\right)dy. \end{aligned}$$

Using Fubini's theorem, we can state:

$$\int_a^b \int_c^d f(x,y)\,dydx = \frac{1}{2}(b-a)\left[\int_c^d f(a,y)\,dy + \int_c^d f(b,y)\,dy\right] - \int_a^b \left(x - \frac{a+b}{2}\right)\left(\int_c^d \frac{\partial f(x,y)}{\partial x}dy\right)dx. \tag{2.4}$$

By the identity (2.2), we can also state:

$$\int_c^d f(a,y)\,dy = \frac{f(a,c)+f(a,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy, \tag{2.5}$$

$$\int_c^d f(b,y)\,dy = \frac{f(b,c)+f(b,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy, \tag{2.6}$$

and

$$\int_c^d \frac{\partial f(x,y)}{\partial x}dy = \frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right](d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}dy. \tag{2.7}$$

Now, using (2.4) and (2.5)-(2.7), we have successively

$$\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dydx \\
= {} & \frac{1}{2}(b-a)\left[\frac{f(a,c)+f(a,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy\right. \\
&\left. + \frac{f(b,c)+f(b,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy\right] \\
& - \int_a^b \left(x - \frac{a+b}{2}\right)\left[\frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right](d-c)\right. \\
&\left. - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}dy\right]dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c) \\
&\quad -\frac{1}{2}(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy \\
&\quad -\frac{1}{2}(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy \\
&\quad -\frac{1}{2}(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,c)}{\partial x}dx \\
&\quad -\frac{1}{2}(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f(x,d)}{\partial x}dx \\
&\quad +\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx
\end{aligned}
$$

and the identity (2.1) is proved. □

The following corollary holds:

Corollary 3. *Let $f,g:[a,b]\to\mathbb{R}$ be differentiable on (a,b) and such that f', $g'\in L_1(a,b)$. Then we have the equality:*

$$
\begin{aligned}
& (b-a)\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx \qquad (2.8)\\
&= \frac{[f(a)-f(b)][g(a)-g(b)]}{4}(b-a)^2 \\
&\quad +\int_a^b\left(x-\frac{a+b}{2}\right)\left(\frac{g(a)+g(b)}{2}-g(x)\right)f'(x)\,dx \\
&\quad +\int_a^b\left(x-\frac{a+b}{2}\right)\left(\frac{f(a)+f(b)}{2}-f(x)\right)g'(x)\,dx \\
&\quad -\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\int_a^b\left(x-\frac{a+b}{2}\right)g'(x)\,dx.
\end{aligned}
$$

Proof. The above identity can be proved by direct computation. We give here a proof based on the previous identity (2.1).

Consider the mapping $h:[a,b]^2\to\mathbb{R}$ given by $h(x,y):=(f(x)-f(y))(g(x)-g(y))$ and write the equality (2.1) for h and the interval $[a,b]^2$ to get

$$
\begin{aligned}
& \int_a^b\int_a^b h(x,y)\,dxdy+\frac{1}{2}(b-a)\int_a^b\left(y-\frac{a+b}{2}\right)\left[\frac{\partial h(a,y)}{\partial y}+\frac{\partial h(b,y)}{\partial y}\right]dy \qquad (2.9)\\
&\quad +\frac{1}{2}(b-a)\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\partial h(x,a)}{\partial x}+\frac{\partial h(x,b)}{\partial x}\right]dx \\
&= \frac{h(a,a)+h(a,b)+h(b,a)+h(b,b)}{4}(b-a)^2 \\
&\quad +\int_a^b\int_a^b\left(x-\frac{a+b}{2}\right)\left(y-\frac{a+b}{2}\right)\frac{\partial^2 h(x,y)}{\partial x\partial y}dxdy.
\end{aligned}
$$

Note that

$$\begin{aligned}\frac{\partial h(x,y)}{\partial x} &= f'(x)(g(x)-g(y))+g'(x)(f(x)-f(y))\\ \frac{\partial h(x,y)}{\partial y} &= f'(y)(g(y)-g(x))+g'(y)(f(y)-f(x))\end{aligned}$$

and then

$$\begin{aligned}&\frac{1}{2}\left[\frac{\partial h(a,y)}{\partial y}+\frac{\partial h(b,y)}{\partial y}\right]\\ =\ & f'(y)\left(g(y)-\frac{g(a)+g(b)}{2}\right)+g'(y)\left(f(y)-\frac{f(a)+f(b)}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\frac{\partial h(x,a)}{\partial x}+\frac{\partial h(x,b)}{\partial x}\right]\\ =\ & f'(x)\left(g(x)-\frac{g(a)+g(b)}{2}\right)+g'(x)\left(f(x)-\frac{f(a)+f(b)}{2}\right).\end{aligned}$$

In addition, we note that

$$\frac{\partial^2 h(x,y)}{\partial x\partial y}=-f'(x)g'(y)-f'(y)g'(x).$$

We have

$$\begin{aligned}I\ :\ &= \frac{1}{2}\int_a^b\left(y-\frac{a+b}{2}\right)\left[\frac{\partial h(a,y)}{\partial y}+\frac{\partial h(b,y)}{\partial y}\right]dy\\ &= \int_a^b\left(y-\frac{a+b}{2}\right)\left[f'(y)\left(g(y)-\frac{g(a)+g(b)}{2}\right)\right.\\ &\quad\left.+g'(y)\left(f(y)-\frac{f(a)+f(b)}{2}\right)\right]dy\\ &= \int_a^b\left(x-\frac{a+b}{2}\right)\left(g(x)-\frac{g(a)+g(b)}{2}\right)f'(x)\,dx\\ &\quad+\int_a^b\left(x-\frac{a+b}{2}\right)\left(f(x)-\frac{f(a)+f(b)}{2}\right)g'(x)\,dx\end{aligned}$$

and, similarly,

$$\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\partial h(x,a)}{\partial x}+\frac{\partial h(x,b)}{\partial x}\right]dx=I.$$

On the other hand,

$$\frac{h(a,a)+h(a,b)+h(b,a)+h(b,b)}{4}=\frac{(f(a)-f(b))(g(a)-g(b))}{2}$$

and

$$\begin{aligned}
& \int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right) \frac{\partial^2 h(x,y)}{\partial x \partial y} dxdy \\
= & -\int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right) \left[f'(x) g'(y) + f'(y) g'(x)\right] dxdy \\
= & -2 \int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right) f'(x) g'(y) \, dxdy \\
= & -2 \int_a^b \left(x - \frac{a+b}{2}\right) f'(x) \, dx \int_a^b \left(x - \frac{a+b}{2}\right) g'(x) \, dx.
\end{aligned}$$

Now, by (2.9) (dividing by 2), we get the identity:

$$\begin{aligned}
& \frac{1}{2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) \, dxdy \\
& + \int_a^b \left(x - \frac{a+b}{2}\right)\left(g(x) - \frac{g(a)+g(b)}{2}\right) f'(x) \, dx \\
& + \int_a^b \left(x - \frac{a+b}{2}\right)\left(f(x) - \frac{f(a)+f(b)}{2}\right) g'(x) \, dx \\
= & \frac{(f(a) - f(b))(g(a) - g(b))}{4} (b-a)^2 \\
& - \int_a^b \left(x - \frac{a+b}{2}\right) f'(x) \, dx \int_a^b \left(x - \frac{a+b}{2}\right) g'(x) \, dx.
\end{aligned}$$

As it is well known that

$$\begin{aligned}
& \frac{1}{2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) \, dxdy \\
= & (b-a) \int_a^b f(x) g(x) \, dx - \int_a^b f(x) \, dx \int_a^b g(x) \, dx,
\end{aligned}$$

the inequality (2.8) is completely proved. □

3. Some Integral Inequalities for $\|\cdot\|_\infty$ −Norm

The following inequality holds.

Theorem 4. *Let f be as in Theorem 3 and assume that*

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty.$$

Then we have the estimation

$$\begin{aligned}(3.1)\quad & \left|\int_a^b\int_c^d f(x,y)\,dydx+(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)f_2(y)\,dy\right.\\ & +(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)f_1(x)\,dx\\ & \left.-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right|\\ \leq\ & \frac{1}{16}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty,\end{aligned}$$

where

$$f_1(x):=\frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x}+\frac{\partial f(x,d)}{\partial x}\right],\ x\in[a,b]$$

and

$$f_2(y):=\frac{1}{2}\left[\frac{\partial f(a,y)}{\partial y}+\frac{\partial f(b,y)}{\partial y}\right],\ y\in[c,d].$$

Proof. Using the identity (2.1) and the properties of the integral, we can state:

$$\begin{aligned}& \left|\int_a^b\int_c^d f(x,y)\,dydx+(b-a)\int_c^d\left(y-\frac{c+d}{2}\right)f_2(y)\,dy\right.\\ & +(d-c)\int_a^b\left(x-\frac{a+b}{2}\right)f_1(x)\,dx\\ & \left.-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right|\\ \leq\ & \int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|dydx\\ \leq\ & \left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\int_a^b\left|x-\frac{a+b}{2}\right|dx\int_c^d\left|y-\frac{c+d}{2}\right|dy\\ =\ & \left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\frac{(b-a)^2}{4}\cdot\frac{(d-c)^2}{4}\end{aligned}$$

and the inequality (3.1) is proved. □

Another inequality which employs the $\|\cdot\|_\infty$ −norm of f_1 and f_2 which can be useful in practice is embodied in the following theorem:

Theorem 5. *Let f be as in Theorem 4 and assume that*

$$\|f_1\|_\infty:=\sup_{x\in[a,b]}|f_1(x)|<\infty,\ \ \|f_2\|_\infty:=\sup_{x\in[a,b]}|f_2(x)|<\infty.$$

Then

$$\left|\int_a^b\int_c^d f(x,y)\,dydx-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \tag{3.2}$$
$$\leq \frac{1}{4}(b-a)(d-c)\times\left[(b-a)\|f_1\|_\infty+(d-c)\|f_2\|_\infty+\frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\right].$$

Proof. As in Theorem 4, we have, by the identity (2.1) that

$$\left|\int_a^b\int_c^d f(x,y)\,dydx-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right|$$
$$\leq (b-a)\int_c^d\left|y-\frac{c+d}{2}\right||f_2(y)|\,dy+(d-c)\int_a^b\left|x-\frac{a+b}{2}\right||f_1(x)|\,dx$$
$$+\int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|dydx$$
$$\leq (b-a)\|f_2\|_\infty\frac{(d-c)^2}{4}+(d-c)\|f_1\|_\infty\frac{(b-a)^2}{4}+\frac{(b-a)^2}{4}\cdot\frac{(d-c)^2}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty$$
$$=\frac{1}{4}(b-a)(d-c)\times\left[(d-c)\|f_2\|_\infty+(b-a)\|f_1\|_\infty+\frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\right].$$

Hence, the proof is completed. □

Remark 1. *If we know that*

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty:=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial f(x,y)}{\partial x}\right|<\infty$$

and

$$\left\|\frac{\partial f}{\partial y}\right\|_\infty:=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial f(x,y)}{\partial y}\right|<\infty,$$

then, obviously

$$\|f_1\|_\infty\leq\left\|\frac{\partial f}{\partial x}\right\|_\infty,\ \|f_2\|_\infty\leq\left\|\frac{\partial f}{\partial y}\right\|_\infty$$

and by (3.2) we deduce

$$\left|\int_a^b\int_c^d f(x,y)\,dydx-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \tag{3.3}$$
$$\leq\frac{1}{4}(b-a)(d-c)\times\left[(b-a)\left\|\frac{\partial f}{\partial x}\right\|_\infty+(d-c)\left\|\frac{\partial f}{\partial y}\right\|_\infty+\frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_\infty\right].$$

Now, let us recall Grüss' inequality:

If $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$m_1 \le f(x) \le M_1,\ \ m_2 \le g(x) \le M_2 \text{ for all } x\in[a,b], \tag{3.4}$$

then we have the inequality:

$$\begin{aligned}\left|(b-a)\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right| \\ \le \frac{1}{4}(M_2-m_2)(M_1-m_1)(b-a)^2\end{aligned} \tag{3.5}$$

and the constant $\frac{1}{4}$ is the best possible one.

Using Corollary 3, we can state a similar result.

Theorem 6. *Let f,g be as in Corollary 3 and assume that $f',\ g',\ f,g\in L_\infty[a,b]$. Then we have the inequality:*

$$\begin{aligned}&\left|(b-a)\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx\right| \\ &\le \frac{1}{4}(b-a)^2\left[\left|(f(a)-f(b))(g(a)-g(b))\right|+\left\|\frac{g(a)+g(b)}{2}-g\right\|\|f'\|_\infty\right. \\ &\left.+\left\|\frac{f(a)+f(b)}{2}-f\right\|_\infty\|g'\|_\infty+\frac{(b-a)^2}{4}\|f'\|_\infty\|g'\|_\infty\right],\end{aligned} \tag{3.6}$$

where, by $\left\|\frac{g(a)+g(b)}{2}-g\right\|_\infty$, we understand

$$\left\|\frac{g(a)+g(b)}{2}-g\right\|_\infty=\sup_{x\in[a,b]}\left|\frac{g(a)+g(b)}{2}-g(x)\right|.$$

Proof. Using the identity (2.8) and the properties of modulus, we have

$$\begin{aligned}&\left|(b-a)\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right| \\ &\le \frac{(b-a)^2}{4}\left|(f(a)-f(b))(g(a)-g(b))\right| \\ &+\int_a^b\left|x-\frac{a+b}{2}\right|\left|\frac{g(a)+g(b)}{2}-g(x)\right||f'(x)|\,dx \\ &+\int_a^b\left|x-\frac{a+b}{2}\right|\left|\frac{f(a)+f(b)}{2}-f(x)\right||g'(x)|\,dx \\ &+\int_a^b\left|x-\frac{a+b}{2}\right||f'(x)|\,dx\int_a^b\left|x-\frac{a+b}{2}\right||g'(x)|\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(b-a)^2}{4}\left|\left(f(a)-f(b)\right)\left(g(a)-g(b)\right)\right| \\
&\quad +\left\|\frac{g(a)+g(b)}{2}-g\right\|\left\|f'\right\|_\infty \int_a^b \left|x-\frac{a+b}{2}\right| dx \\
&\quad +\left\|\frac{f(a)+f(b)}{2}-f\right\|_\infty \left\|g'\right\|_\infty \int_a^b \left|x-\frac{a+b}{2}\right| dx \\
&\quad +\left\|f'\right\|_\infty \left\|g'\right\|_\infty \left(\int_a^b \left|x-\frac{a+b}{2}\right| dx\right)^2
\end{aligned}
$$

and the inequality (3.6) is proved. □

4. Some Integral Inequalities for $\|\cdot\|_p$ −Norm, $p \in [1,\infty)$.

The following result also holds.

Theorem 7. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 3 and assume that*

$$
\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_p := \left(\int_a^b \int_c^d \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right|^p dydx\right)^{\frac{1}{p}} < \infty, \quad p \in [1,\infty). \tag{4.1}
$$

Then we have the estimate

$$
\begin{aligned}
&\left|\int_a^b \int_c^d f(x,y)\, dydx + (b-a)\int_c^d \left(y-\frac{c+d}{2}\right) f_2(y)\, dy\right. \\
&+ (d-c)\int_a^b \left(x-\frac{a+b}{2}\right) f_1(x)\, dx \\
&\left. - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right| \\
&\leq \begin{cases} \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_1 & \text{if } \frac{\partial^2 f}{\partial x \partial y} \in L_1[a,b]; \\[2ex] \left[\frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\right]\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_p, & \text{if } \frac{\partial^2 f}{\partial x \partial y} \in L_p[a,b] \text{ and} \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases}
\end{aligned} \tag{4.2}
$$

where

$$
f_1(x) = \frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right], \quad x \in [a,b]
$$

and

$$
f_2(y) = \frac{1}{2}\left[\frac{\partial f(a,y)}{\partial y} + \frac{\partial f(b,y)}{\partial y}\right], \quad y \in [c,d].
$$

Proof. It is easy to see that

$$\begin{aligned}
&\int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|dydx\\
\leq\;& \sup_{(x,y)\in[a,b]\times[c,d]}\left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\int_a^b\int_c^d\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|dydx\\
=\;& \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_1
\end{aligned}$$

and, by Hölder's inequality for double integrals, ($p>1$, $\frac{1}{p}+\frac{1}{q}=1$), we have

$$\begin{aligned}
&\int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right|\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|dydx\\
\leq\;& \left(\int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|^q\left|y-\frac{c+d}{2}\right|^q dydx\right)^{\frac{1}{q}}\left(\int_a^b\int_c^d\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|^p dydx\right)^{\frac{1}{p}}\\
=\;& \left[\frac{(b-a)^{1+\frac{1}{q}}}{2^q(q+1)}\right]^{\frac{1}{q}}\left[\frac{(d-c)^{1+\frac{1}{q}}}{2^q(q+1)}\right]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_p\\
=\;& \frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_p.
\end{aligned}$$

Using the representation (2.1) we get the desired result. □

Remark 2. *Similar results to those in Theorems 5 and 6 may be stated, but we omit the details.*

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