CHEBYCHEV FUNCTIONAL BOUNDS USING OSTROWSKI SEMINORMS

P. CERONE AND S.S. DRAGOMIR

Abstract. Bounds are obtained for the Chebychev functional using what is termed as the *Ostrowski seminorm* which is related to an inequality developed by Ostrowski. The Ostrowski seminorm is also compared to the $\Delta-$seminorm introduced in earlier work by the authors.

1. The Ostrowski Seminorm

For a measurable function $f:[a,b]\to\mathbb{R}$, we define the functional

$$\|f\|_p^\theta := \left(\int_a^b\int_a^b\left|\frac{f(x)-f(y)}{x-y}\right|^p dxdy\right)^{\frac{1}{p}},\ p\geq 1 \tag{1.1}$$

provided the above integral exists and is finite.

It is obvious that

(i) $\|f\|_p^\theta\geq 0$ and $\|f\|_p^\theta=0$ implies $f=$ constant a.e. on $[a,b]$;
(ii) $\|\alpha f\|_p^\theta=|\alpha|\,\|f\|_p^\theta$ for $\alpha\in\mathbb{R}$,
(iii) $\|f+g\|_p^\theta\leq\|f\|_p^\theta+\|g\|_p^\theta$,

showing that $\|\cdot\|_p^\theta$ is a seminorm on the space of all measurable functions for which the integral

$$\int_a^b\int_a^b\left|\frac{f(x)-f(y)}{x-y}\right|^p dxdy \tag{1.2}$$

is finite.

We will call $\|\cdot\|_p^\theta$ the *Ostrowski seminorm*, because of the following fundamental inequality established by Ostrowski in [15]

$$\int_a^b\int_a^b\left|\frac{f(x)-f(y)}{x-y}\right|^p dxdy\leq(b-a)\log 4\int_a^b|f'(t)|^p\,dt, \tag{1.3}$$

where $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$ and $p\geq 1$. The constant $\log 4$ is best for $p=1$. Fink [12] applies (1.3) and a generalisation to averages of divided differences.

Consequently, if we use the classical notations for Lebesgue norms, we may state the following theorem.

Date: September 12, 2000.
1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 26D99.
Key words and phrases. Ostrowski and Delta seminorms, Chebyshev functional, Bounds.

Theorem 1. *(Ostrowski) Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $p\geq 1$. Then*

$$\|f\|_p^{\theta}\leq (b-a)^{\frac{1}{p}}\left[\log 4\right]^{\frac{1}{p}}\|f'\|_p. \tag{1.4}$$

The constant $\log 4$ is sharp for $p=1$.

2. Comparing the Ostrowski Seminorm with $\Delta-$Seminorms

For $f\in L_p[a,b]$ $(p\in[1,\infty))$ we can define the functional (see also [2])

$$\|f\|_p^{\Delta}:=\left(\int_a^b\int_a^b |f(t)-f(s)|^p\,dtds\right)^{\frac{1}{p}} \tag{2.1}$$

and for $f\in L_\infty[a,b]$, we can define

$$\|f\|_\infty^{\Delta}:=ess\sup_{(t,s)\in[a,b]^2}|f(t)-f(s)|. \tag{2.2}$$

If we consider $f_\Delta:[a,b]^2\to\mathbb{R}$,

$$f_\Delta(t,s)=f(t)-f(s),$$

then, obviously,

$$\|f\|_p^{\Delta}=\|f_\Delta\|_p,\quad g\in[1,\infty], \tag{2.3}$$

where $\|\cdot\|_p$ are the usual Lebesgue $p-$norms on $[a,b]^2$.

Using the properties of the Lebesgue $p-$norms, we may deduce the following semi-norm properties of $\|\cdot\|_p^{\Delta}$:

(i) $\|f\|_p^{\Delta}\geq 0$ for $f\in L_p[a,b]$ and $\|f\|_p^{\Delta}=0$ implies that $f=c$ (c is a constant) a.e. on $[a,b]$;
(ii) $\|f+g\|_p^{\Delta}\leq\|f\|_p^{\Delta}+\|g\|_p^{\Delta}$ if $f,g\in L_p[a,b]$,
(iii) $\|\alpha f\|_p^{\Delta}=|\alpha|\,\|f\|_p^{\Delta}$ if $f\in L_p[a,b]$, $\alpha\in\mathbb{R}$.

We note that if $p=2$, then

$$\begin{aligned}\|f\|_2^{\Delta} &= \left(\int_a^b\int_a^b (f(t)-f(s))^2\,dtds\right)^{\frac{1}{2}}\\ &= \sqrt{2}\left[(b-a)\|f\|_2^2-\left(\int_a^b f(t)\,dt\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^{\Delta}$ in terms of $\|f'\|_p$.

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$.*

(i) *If $p \in [1, \infty)$, then we have the inequality:*

$$\|f\|_p^{\Delta} \leq \begin{cases} \dfrac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty & \text{if} \quad f' \in L_\infty [a,b] \\[2ex] \dfrac{(2\beta^2)^{\frac{1}{\beta}} (b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha [a,b], \\ & \quad \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (b-a)^{\frac{2}{p}} \|f'\|_1 . \end{cases} \tag{2.4}$$

(ii) *If $p = \infty$, then we have the inequality:*

$$\|f\|_p^{\Delta} \leq \begin{cases} (b-a) \|f'\|_\infty & \text{if} \quad f' \in L_\infty [a,b] \\[2ex] (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha [a,b], \\ & \quad \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1 . \end{cases} \tag{2.5}$$

For a short proof of the above inequalities, see for example [2].

The following result in comparing the $\Delta-$seminorm with the Ostrowski-seminorm holds.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a measurable function such that $\|f\|_p^{\theta}$ $(p \geq 1)$ exists. Then we have the inequality:*

$$\|f\|_p^{\Delta} \leq \frac{2^{\frac{1}{p\beta}} (b-a)^{1+\frac{2}{p\beta}}}{[(p\beta+1)(p\beta+2)]^{\frac{1}{p\beta}}} \|f\|_{p\alpha}^{\theta}, \quad \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1. \tag{2.6}$$

Proof. We have that

$$\begin{aligned} \|f\|_p^{\Delta} &= \left(\int_a^b \int_a^b |f(x) - f(y)|^p \, dxdy \right)^{\frac{1}{p}} \\ &= \left(\int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x-y} \right|^p \cdot |x-y|^p \, dxdy \right)^{\frac{1}{p}} \\ &\leq \left(\int_a^b \int_a^b \left| \frac{f(x) - f(y)}{x-y} \right|^{p\alpha} dxdy \right)^{\frac{1}{p\alpha}} \left(\int_a^b \int_a^b |x-y|^{p\beta} \, dxdy \right)^{\frac{1}{p\beta}} \\ &= \|f\|_{p\alpha}^{\theta} \left[\frac{2(b-a)^{p\beta+2}}{(p\beta+1)(p\beta+2)} \right]^{\frac{1}{p\beta}} = \frac{2^{\frac{1}{p\beta}} (b-a)^{1+\frac{2}{p\beta}}}{[(p\beta+1)(p\beta+2)]^{\frac{1}{p\beta}}} \|f\|_{p\alpha}^{\theta} \end{aligned}$$

which proves (2.6). ∎

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the inequality:*

$$\|f\|_p^{\Delta} \leq \frac{2^{\frac{1}{p\beta}} (\log 4)^{\frac{1}{p\alpha}} (b-a)^{1+\frac{2}{p\beta}+\frac{1}{p\alpha}}}{[(p\beta+1)(p\beta+2)]^{\frac{1}{p\beta}}} \|f'\|_{p\alpha}, \tag{2.7}$$

for $p \geq 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha > 1$.

Proof. Using Ostrowski's inequality (1.3), we have from (1.1)

$$\|f\|_{p\alpha}^{\theta} \leq (\log 4)^{\frac{1}{p\alpha}} (b-a)^{\frac{1}{p\alpha}} \|f'\|_{p\alpha}$$

and then, by (2.6) we get (2.7). ∎

3. Some Bounds for the Chebychev Functional

For two measurable functions $f, g : [a,b] \to \mathbb{R}$, we define the Chebychev-functional as follows

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{3.1}$$

It is well known that if f, g are synchronous, i.e.,

$$(f(t)-f(s))(g(t)-g(s)) \geq 0 \text{ for a.e. } t,s \in [a,b],$$

then

$$T(f,g;a,b) \geq 0 \tag{3.2}$$

which is called in the literature the Chebychev inequality for synchronous functions.

If $m \leq f \leq M$, $n \leq g \leq N$ a.e. on $[a,b]$, m, M, n, N are real functions, then the following Grüss inequality holds (see for example [13] or [16] and [14])

$$|T(f,g;a,b)| \leq \frac{1}{4}(M-m)(N-n). \tag{3.3}$$

The constant $\frac{1}{4}$ in (3.3) is best possible.

If $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous and $f', g' \in L_\infty[a,b]$, that is,

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < 0$$

and similarly $\|g'\|_\infty < \infty$, then we can state another result by Chebychev

$$|T(f,g;a,b)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{3.4}$$

where the constant $\frac{1}{12}$ is the best possible.

If we assume that the derivatives $f', g' \in L_2[a,b]$, then we may point out the following bound in terms of the euclidean norm due to Lupaş:

$$|T(f,g;a,b)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a) \tag{3.5}$$

for which the constant $\frac{1}{\pi^2}$ is sharp.

For these and related work involving bounds of the Chebychev functional, see the references [1] – [10].

Now, we point out some bounds for the modulus of Chebychev's functional in terms of the Ostrowski seminorm introduced above.

Theorem 4. *Assume that $f, g : [a,b] \to \mathbb{R}$ are measurable and $p, r, q > 0$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$, $p > 1$. Then*

$$|T(f,g;a,b)| \leq \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(2p+1)(2p+2)]^{\frac{1}{p}}}\|f\|_q^\theta \|g\|_r^\theta, \tag{3.6}$$

provided that all the integrals involved exist.

Proof. Using Korkine's identity for integrals, i.e.,

$$T(f,g;a,b)=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds$$

we have, by Hölder's inequality for p,q,r with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $p>1$, that

$$\begin{aligned}
&|T(f,g;a,b)| \\
\leq\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\
=\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)^2\left|\frac{f(t)-f(s)}{t-s}\right|\left|\frac{g(t)-g(s)}{t-s}\right| dtds \\
\leq\;& \frac{1}{2(b-a)^2}\left(\int_a^b\int_a^b (t-s)^{2p}\,dtds\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b \left|\frac{f(t)-f(s)}{t-s}\right|^q dtds\right)^{\frac{1}{q}} \\
&\times\left(\int_a^b\int_a^b \left|\frac{f(t)-f(s)}{t-s}\right|^r dtds\right)^{\frac{1}{r}} \\
=\;& \frac{1}{2(b-a)^2}\left[\frac{2(b-a)^{2p+2}}{(2p+1)(2p+2)}\right]^{\frac{1}{p}}\|f\|_q^{\theta}\,\|g\|_r^{\theta} \\
=\;& \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(2p+1)(2p+2)]^{\frac{1}{p}}}\|f\|_q^{\theta}\,\|g\|_r^{\theta}
\end{aligned}$$

and the theorem is proved. ∎

Using Ostrowski's theorem, we may state the following corollary concerning f' with an upper bound of the modulus of Chebychev's functional in terms of the usual Lebesgue norms of the derivative.

Corollary 2. *Assume that the functions $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous and p,r,q are as in Theorem 2. Then*

$$|T(f,g;a,b)|\leq\frac{2^{\frac{1}{p}-1}(\log 4)^{1-\frac{1}{p}}(b-a)^{1+\frac{1}{p}}}{[(2p+1)(2p+2)]^{\frac{1}{p}}}\|f'\|_q\,\|g'\|_r\,, \tag{3.7}$$

Proof. By Ostrowski's inequality (1.3) we have

$$\|f\|_q^{\theta}\leq(b-a)^{\frac{1}{q}}(\log 4)^{\frac{1}{q}}\|f'\|_q$$

and

$$\|g\|_r^{\theta}\leq(b-a)^{\frac{1}{r}}(\log 4)^{\frac{1}{r}}\|g'\|_r\,.$$

Thus, using (3.6) we deduce (3.7). ∎

Remark 1. *If $p\to1$ in (3.7), then Chebychev's result (3.4) is recaptured.*

Rather than using coarser upper bounds as provided by Corollary 2, for certain functions the Ostrowski functional may be evaluated directly.

Corollary 3. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Further, let $p,q,r>0$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $p>1$, then*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left[\left(\frac{z-a}{b-a}\right)f(a)+\left(\frac{b-z}{b-a}\right)f(b)\right]\right. \\ &\left.-\left(z-\frac{a+b}{2}\right)S(f;a,b)\right| \\ \leq\;& \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}+\frac{2}{q}-1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}}\left\|f'\right\|_r^{\theta} \\ \leq\;& \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}+\frac{2}{q}+\frac{1}{r}-1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}}(\log 4)^{\frac{1}{r}}\left(\left\|f''\right\|_r\right)^r, \end{aligned} \tag{3.8}$$

provided all the integrals involved exist, where

$$S(f;a,b)=\frac{f(b)-f(a)}{b-a},$$

is the secant slope.

Proof. If we take

$$h(t)=\frac{t-z}{b-a},\quad z\in[a,b] \tag{3.9}$$

and associate f with h and g with f' in (3.1), then

$$\begin{aligned} -T(h,f';a,b) &= \frac{1}{b-a}\int_a^b f(t)\,dt-\left[\left(\frac{z-a}{b-a}\right)f(a)\right. \\ &\left.+\left(\frac{b-z}{b-a}\right)f(b)\right]-\left(z-\frac{a+b}{2}\right)S(f;a,b) \end{aligned} \tag{3.10}$$

where a simple integration by parts has been preformed.

Now, from (1.1) and (3.9)

$$\|h\|_q^{\theta}=\left(\int_a^b\int_a^b\left|\frac{h(x)-h(y)}{x-y}\right|^q dxdy\right)^{\frac{1}{q}}=(b-a)^{\frac{2}{q}-1}. \tag{3.11}$$

Hence, using (3.10), (3.11) and (3.6) gives the first inequality in (3.8).

Further, using the Ostrowski inequality (1.3) gives, on utlising (1.1),

$$\left\|f'\right\|_r^{\theta}\leq(b-a)\log 4\int_a^b\left|f''(t)\right|^r dt$$

and thus the second inequality is obtained on noting that

$$\int_a^b\left|f''(t)\right|^r dt=\left(\left\|f''\right\|_r\right)^r,$$

where $\|\cdot\|_r$ is the usual Lebesgue norm. ∎

Remark 2. *We note that the bounds are independent of the point z. If we take $z=\frac{a+b}{2}$, then the standard trapezoidal rule is obtained from the perturbed rule (3.8) on multiplying both sides by $b-a$.*

Corollary 4. *Let $f : [a,b] \to \mathbb{R}$ be a mapping for which $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$ and $p,q,r > 0$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p > 1$ then,*

$$\begin{aligned}
(3.12) \quad & \left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(z-a)^{k+1} f^{(k)}(a) + (-1)^k (b-z)^{k+1} f^{(k)}(b) \right] \right. \\
& \left. - \frac{(z-a)^{n+1} + (-1)^n (b-z)^{n+1}}{(n+1)!} S_n(f;a,b) \right| \\
\leq \; & \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}+1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \cdot \frac{1}{n!} \left(\int_a^b \int_a^b \left| \frac{(z-x)^n - (z-y)^n}{x-y} \right|^q dxdy \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_r^{\theta} \\
\leq \; & \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}+1}}{[(2p+1)(2p+2)]^{\frac{1}{p}} [(n-1)!]^p} \left[\frac{(z-a)^{(n-1)p+1} + (b-z)^{(n-1)p+1}}{(n-1)p+1} \right] \left\| f^{(n)} \right\|_r^{\theta},
\end{aligned}$$

where $S_n(f;a,b) = \frac{f^{(n)}(b) - f^{(n)}(b)}{b-a}$.

Proof. The proof follows closely that of the previous corollary.

Cerone et al. [3] obtained the identity

$$\begin{aligned}
(3.13) \quad \int_a^b f(t)\,dt \;=\; & \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(z-a)^{k+1} f^{(k)}(a) \right. \\
& \left. + (-1)^k (b-z)^{k+1} f^{(k)}(b) \right] + \int_a^b H_n(t) f^{(n)}(t)\,dt
\end{aligned}$$

where

$$(3.14) \qquad H_n(t) = \frac{(z-t)^n}{n!}. \;\; z \in [a,b].$$

Now, if we associate f with H_n and g with $f^{(n)}$ in (3.1), then

$$\begin{aligned}
(3.15) \quad & T\left(H_n, f^{(n)}; a, b\right) \\
= \; & \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(z-a)^{k+1} f^{(k)}(a) \right. \\
& \left. + (-1)^k (b-z)^{k+1} f^{(k)}(b) \right] - \int_a^b H_n(t)\,dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt.
\end{aligned}$$

Further, from (3.14)

$$\int_a^b H_n(t)\,dt = \frac{(z-a)^{n+1} + (-1)^n (b-z)^{n+1}}{(n+1)!}$$

and

$$\frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

giving from (3.1) and (3.15) the left hand side of (3.15). Now, from (3.6) and (3.15) and using (3.14) gives

$$(3.16) \qquad \|H_n\|_q^{\theta} = \frac{1}{n!} \left(\int_a^b \int_a^b \left| \frac{(z-x)^n - (z-y)^n}{x-y} \right|^q dxdy \right)^{\frac{1}{q}}$$

and so the first inequality in (3.12) results.

Now, for the second inequality (3.12) we use Ostrowski's inequality on $H_n(t)$ as defined by (3.14) to give

$$\int_a^b \int_a^b \left|\frac{H_n(x) - H_n(y)}{x-y}\right|^p dxdy \leq (b-a)\log 4 \int_a^b |H_n'(t)|^p dt = (b-a)\log 4 \int_a^b \left|\frac{(z-t)^{n-1}}{(n-1)!}\right|^p dt,$$

which, upon explicit evaluation gives the second inequality. ∎

Remark 3. *The second inequality in (3.12) is coarser than the first. For particular values of n the double integral in the first inequality may be evaluated explicitly. From (3.16),*

$$\|H_2\|_q^\theta = \frac{1}{2!}\left(\int_a^b \int_a^b \left|\frac{(z-x)^2-(z-y)^2}{x-y}\right|^q dxdy\right)^{\frac{1}{q}} = \frac{1}{2}\left(\int_a^b \int_a^b |z-x+z-y|^q dxdy\right)^{\frac{1}{q}}. \tag{3.17}$$

Let

$$J := \int_a^b \int_a^b |z-x+z-y|^q dxdy$$

then

$$\begin{aligned} J &= \int_a^z \int_a^z |z-x+z-y|^q dxdy + \int_z^b \int_a^z |z-x+z-y|^q dxdy \\ &+ \int_a^z \int_z^b |z-x+z-y|^q dxdy + \int_z^b \int_z^b |z-x+z-y|^q dxdy \\ &: = J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.18}$$

Now, it may be verified by some straightforward algebra that

$$D_q(W) = \int_0^W \int_0^W |u-v|^q dudv = \frac{2W^{q+2}}{(q+1)(q+2)} \tag{3.19}$$

and

$$A_q(W) = \int_0^W \int_0^W |u+v|^q dudv = \frac{2\left(2^{q+1}-1\right)W^{q+2}}{(q+1)(q+2)}. \tag{3.20}$$

Thus, by making some obvious transformations, it may be determined that

$$J_1 = A_q(z-a) \quad \text{and} \quad J_4 = A_q(b-z). \tag{3.21}$$

Further, it may be ascertained by symmetry that $J_2 = J_3$ and hence we will consider the evaluation of J_2 in some detail. Now,

$$J_2 = \begin{cases} D_q(z-a) + \int_{z-a}^{b-z} \int_0^{z-a} (v-u)^q dudv, & z < \frac{a+b}{2}, \\ D_q(b-z) + \int_0^{b-z} \int_{b-z}^{z-a} (u-v)^q dudv, & z > \frac{a+b}{2}, \end{cases}$$

and so, after some algebra

$$J_2 = \frac{(z-a)^{q+2} + (b-z)^{q+2} - 2^{q+2}\left|z - \frac{a+b}{2}\right|^{q+2}}{(q+1)(q+2)} \; (= J_3) . \tag{3.22}$$

Combining the results (3.18) – (3.22) into (3.17) gives

$$\|H_2\|_q^\theta = \frac{2^{\frac{2}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} \left[(z-a)^{q+2} + (b-z)^{q+2} - 2\left|z - \frac{a+b}{2}\right|^{q+2} \right]^{\frac{1}{q}} \tag{3.23}$$

and so from (3.12) with $n = 2$

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \left[(z-a) f(a) + (b-z) f(b) + \frac{1}{2!}(z-a)^2 f'(a) - \frac{1}{2!}(b-z)^2 f'(b) \right] \right. \\ & \left. - \frac{(z-a)^3 + (b-z)^3}{3!} \frac{[f'(b) - f'(a)]}{b-a} \right| \\ = & \left| \int_a^b f(t)\,dt - [(z-a) f(a) + (b-z) f(b)] - \frac{(z-a)^2}{6(b-a)} (z - a - 3(b-a)) f'(a) \right. \\ & \left. \left. + \frac{(b-z)^2}{6(b-a)} (b - z - 3(b-a)) f'(b) \right] \right| \\ \leq & \frac{2^{\frac{1}{p}-1} (b-a)^{\frac{2}{p}+1}}{[(2p+1)(2p+2)]^{\frac{1}{p}}} \|H_2\|_q^\theta \|f''\|_r^\theta , \end{aligned}$$

where $\|H_2\|_q^\theta$ is as given by (3.23).

If $z = \frac{a+b}{2}$, then some simplification occurs to give

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)] + \frac{5}{48}(b-a)[f'(b) - f'(a)] \right| \\ \leq & \frac{2^{2\left(\frac{1}{p}+\frac{1}{q}-1\right)} (b-a)^{2\left(\frac{1}{p}+\frac{1}{q}-1\right)}}{[(2p+1)(2p+2)]^{\frac{1}{p}} [(q+1)(q+2)]^{\frac{1}{q}}} \|f''\|_r^\theta . \end{aligned}$$

Remark 4. *Using the definition of the functional (1.1), Ostrowski's inequality (1.3) could be used to obtain upper bounds on the inequalities (3.12) using*

$$\left\| f^{(n)} \right\|_r^\theta \leq (b-a) \log 4 \left(\left\| f^{(n+1)} \right\|_r \right)^r$$

in terms of the usual Lebesgue norm for $f^{(n+1)}$.

References

[1] P. CERONE and S. DRAGOMIR, Generalisations of the Grüss, Chebychev and Lupaş inequalities for integrals over different intervals, *RGMIA Res. Rep. Coll.,* **3**(2) (2000), Article 6.

[2] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Grüss inequality in terms of $\Delta-$seminorms and applications, *RGMIA Res. Rep. Coll.,* **3**(3) (2000), Article 4.

[3] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS and J. ŠUNDE, A new generalisation for the trapezoid formula for $n-$time differentiable mappings and applications, *Demonstratio Mathematica,* **33**(4) (2000), 719-736.

[4] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.

[5] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43-47.
[6] S.S. DRAGOMIR, Better bounds in some Ostrowski-Grüss type inequalities, *RGMIA Res. Rep. Coll.,* **3**(1) (2000), Article 3.
[7] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.
[8] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4), (2000), 397-415.
[9] S.S. DRAGOMIR and G.L. BOOTH, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Mathematical Communications,* **5** (2000), 117-126.
[10] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4)(1998), 286-292.
[11] A.M. FINK, A treatise on Grüss' inequality, *submitted*.
[12] A.M. FINK, Inequalities for averages and divided differences, *Math. Ineq. & Appl.,* **3** (2000), 327-336.
[13] G. GRÜSS, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.
[14] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[15] A.M. OSTROWSKI, On some integral inequalities II, *Basel Mathematical Notes,* **24** (1965), 1-11.
[16] J. PEČARIĆ, F. PROSCHAN and Y. TONG, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, San Diego, 1992.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: pc@matilda.vu.edu.au
URL: http://sci.vu.edu.au/staff/peterc.html

E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html