SOME INEQUALITIES FOR THE KULLBACK-LEIBLER AND χ^2-DISTANCES IN INFORMATION THEORY AND APPLICATIONS

S. S. DRAGOMIR AND V. GLUŠČEVIĆ

Abstract. Inequalities for the Kullback-Leibler and χ^2-distances and applications for Shannon's entropy and mutual information are given.

1. Introduction

Let $p(x), q(x), x \in \mathfrak{X}, card(\mathfrak{X}) < \infty$, be two probability mass functions. Define the *Kullback-Leibler distance* (see [1] or [2]) by

$$KL(p,q) := \sum_{x\in\mathfrak{X}} p(x) \log \frac{p(x)}{q(x)},$$

the χ^2-*distance* (see for example [3]) by

$$D_{\chi^2}(p,q) := \sum_{x\in\mathfrak{X}} \frac{p^2(x) - q^2(x)}{q(x)}$$

and the *variation distance* (see for example [3]) by

$$V(p,q) := \sum_{x\in\mathfrak{X}} |p(x) - q(x)|.$$

The following theorem is of fundamental importance in Information Theory [4, p. 26].

Theorem 1. *(Information Inequality). Under the above assumptions for p and q, we have*

$$KL(p,q) \geq 0,$$

with equality if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

This inequality can be improved as follows (see [4, p. 300]):

Theorem 2. *Let p, q be as above. Then*

$$KL(p,q) \geq \frac{1}{2} V^2(p,q) \geq 0, \tag{1.1}$$

with equality if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

In [5] (see also [6]), the authors proved the following counterpart of (1.1).

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Theorem 3. *Let $p(x), q(x) > 0, x \in \mathfrak{X}$ be two probability mass functions. Then*

$$D_{\chi^2}(p,q) \geq KL(p,q) \geq 0, \tag{1.2}$$

with equality if and only if $p(x) = q(x)$, $x \in \mathfrak{X}$.

In the same paper [6], the authors applied (1.2) for Shannon's entropy, mutual information, etc....

In Section 2 of the present paper, we provide improvement of result (1.2), in Section 3 we apply this result to Entropy and in Section 4 to mutual information in the same manner as in [6].

2. An Improved Inequality and Related Results

The following result holds.

Theorem 4. *Let $p(x), q(x) > 0, x \in \mathfrak{X}$ be two probability mass functions. We have the inequality*

$$0 \leq KL(p,q) \leq \log\left[D_{\chi^2}(p,q) + 1\right] \leq D_{\chi^2}(p,q). \tag{2.1}$$

Equality holds in (2.1) if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

Proof. We use Jensen's discrete inequality

$$f\left(\sum_{x\in\mathfrak{X}} p(x)\, t(x)\right) \leq \sum_{x\in\mathfrak{X}} p(x)\, f(t(x)), \tag{2.2}$$

provided that f is convex on a given interval I, $t(x) \in I$ for all $x \in \mathfrak{X}$ and $p(x)$ is a probability mass function on $\mathfrak{X}$.

Choosing $f(s) = -\log s,\ s > 0$ and $t(x) = \frac{p(x)}{q(x)}$, we obtain in (2.2)

$$-\log\left(\sum_{x\in\mathfrak{X}} p(x)\frac{p(x)}{q(x)}\right) \leq -\sum_{x\in\mathfrak{X}} p(x)\log\left(\frac{p(x)}{q(x)}\right),$$

which is the first inequality in (2.1).

Equality holds in the first part of (2.1) if and only if $\frac{p(x)}{q(x)} = \frac{p(y)}{q(y)}$ for all $x, y \in \mathfrak{X}$, which is equivalent to $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

For the second inequality, we use the following elementary inequality

$$\log t \leq t - 1 \qquad \text{for all } t > 0$$

with equality if and only if $t = 1$.

Equality holds in the second part of (2.1) if and only if $D_{\chi^2}(p,q) = 0$, which holds if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$. □

In 1948, B.L. Kantorović (see for example [7]) proved the following inequality for sequences of real numbers

$$\sum_{k=1}^{n} r_k u_k^2 \sum_{k=1}^{n} \frac{1}{r_k} u_k^2 \leq \frac{1}{4}\left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right)^2 \left(\sum_{k=1}^{n} u_k^2\right)^2,$$

where

$$0 < m \leq r_k \leq M < \infty \ \text{ for } k = 1, ..., n.$$

Using this result, we derived the following upper bound for the χ^2-distance (see also [6]).

Lemma 1. *Let $p(x), q(x) > 0, x \in \mathfrak{X}$ be two probability mass functions. Define $r(x) := \frac{p(x)}{q(x)}, x \in \mathfrak{X}$ and assume that*

$$0 < r \le r(x) < R < \infty \quad \text{for all } x \in \mathfrak{X}.$$

Then we have the inequality

$$0 \le D_{\chi^2}(p,q) \le \frac{(R-r)^2}{4rR}. \tag{2.3}$$

Equality holds in (2.3) if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

Proof. Using the Kantorović inequality for $r_k = r(x)$ and $u_k = \sqrt{p(x)}$, we can state that

$$\sum_{x\in\mathfrak{X}} p(x) r(x) \sum_{x\in\mathfrak{X}} p(x) \frac{1}{r(x)} \le \frac{1}{4}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right)^2 \left(\sum_{x\in\mathfrak{X}} p(x)\right)^2,$$

which is equivalent to

$$\sum_{x\in\mathfrak{X}} \frac{p^2(x)}{q(x)} \le \frac{1}{4}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right)^2,$$

from where we deduce

$$\begin{aligned} D_{\chi^2}(p,q) &\le \frac{1}{4}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right)^2 - 1 \\ &= \frac{1}{4}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{r}{R}}\right)^2 \end{aligned}$$

and the inequality (2.3) is proved.

The case of equality holds in (2.3) by the fact that in the Kantorović inequality, we have equality if and only if $r_k = 1$ for all $k \in \{1, ..., n\}$. □

The following theorem holds.

Theorem 5. *Let p, q, r, R be as in Lemma 1. Then we have the inequality*

$$0 \le KL(p,q) \le \log\left[\frac{(R-r)^2}{4rR} + 1\right] \le \frac{(R-r)^2}{4rR}. \tag{2.4}$$

Equality holds if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

Proof. Using the first inequality in (2.1) and (2.3), we have

$$KL(p,q) \le \log\left[D_{\chi^2}(p,q) + 1\right] \le \log\left[\frac{(R-r)^2}{4rR} + 1\right].$$

The last inequality in (2.4) follows by the elementary inequality $\log(u+1) \le u$, $u \ge 0$ with equality if and only if $u = 0$. □

Remark 1. *The inequality (2.4) improves the result*

$$0 \le KL(p,q) \le \frac{(R-r)^2}{4rR}, \tag{2.5}$$

which was proved in [6]. *We also note that (2.5) was proved independently by M. Matić in* [8] *using another technique based on Grüss' discrete inequality.*

The following corollary holds.

Corollary 1. *Let p, q, r, R be as in Lemma 1. Define*

$$S := \frac{R}{r} \ (\geq 1).$$

If $\varepsilon > 0$ and

$$S \leq 2e^{\varepsilon} - 1 + 2\sqrt{e^{\varepsilon}\left(e^{\varepsilon} - 1\right)}, \tag{2.6}$$

then we have the inequality

$$0 \leq KL(p, q) \leq \varepsilon.$$

Proof. Observe that for a given $\varepsilon > 0$, the inequality

$$\log\left(\frac{(R-r)^2}{4rR} + 1\right) \leq \varepsilon$$

is equivalent to

$$\frac{(R-r)^2}{4rR} \leq e^{\varepsilon} - 1,$$

i.e.,

$$R^2 - 2\left[1 + 2\left(e^{\varepsilon} - 1\right)\right] rR + r^2 \leq 0. \tag{2.7}$$

Dividing (2.7) by $r^2 > 0$, we obtain

$$S^2 - 2\left[1 + 2\left(e^{\varepsilon} - 1\right)\right] S + 1 \leq 0,$$

which is clearly equivalent to

$$S \in \left[2e^{\varepsilon} - 1 - 2\sqrt{e^{\varepsilon}\left(e^{\varepsilon} - 1\right)}, 2e^{\varepsilon} - 1 + 2\sqrt{e^{\varepsilon}\left(e^{\varepsilon} - 1\right)}\right]. \tag{2.8}$$

Furthermore, as $S \geq 1$, then (2.8) follows by (2.6) and then (2.6) implies, by (2.4), that $KL(p, q) \leq \varepsilon$. □

The following result is well known in the literature as the Diaz-Metcalf inequality for real numbers (see for example [7, p. 61]):

$$\sum_{k=1}^{n} p_k b_k^2 + mM \sum_{k=1}^{n} p_k a_k^2 \leq (m + M) \sum_{k=1}^{n} p_k a_k b_k \tag{2.9}$$

provided that $m \leq \frac{b_k}{a_k} \leq M$ for $k = 1, ..., n$ and $p_k > 0$ with $\sum_{k=1}^{n} p_k = 1$.

The equality holds in (2.9) if and only if either $b_k = ma_k$ or $b_k = Ma_k$ for $k \in \{1, ..., n\}$.

The following lemma holds (see also [6]).

Lemma 2. *Let p, q, r, R be as in Lemma 1. Then we have the inequality*

$$0 \leq D_{\chi^2}(p, q) \leq (1 - r)(R - 1) \leq \frac{1}{4}(R - r)^2. \tag{2.10}$$

Equality holds if and only if $p(x) = q(x)$ for all $x \in \mathfrak{X}$.

Proof. Define

$$b(x) = \sqrt{\frac{p(x)}{q(x)}},\ a(x) = \sqrt{\frac{q(x)}{p(x)}},\ x \in \mathfrak{X}.$$

Then

$$\frac{b(x)}{a(x)} = \frac{p(x)}{q(x)} = r(x) \in [r, R] \subset (0, \infty) \quad \text{for all } x \in \mathfrak{X}.$$

Applying the Diaz-Metcalf inequality, we deduce

$$\begin{aligned} &\sum_{x\in\mathfrak{X}} p(x)\left(\sqrt{\frac{p(x)}{q(x)}}\right)^2 + Rr\sum_{x\in\mathfrak{X}} p(x)\left(\sqrt{\frac{q(x)}{p(x)}}\right)^2 \\ \leq\ & (r+R)\sum_{x\in\mathfrak{X}} \sqrt{\frac{p(x)}{q(x)}}\cdot\sqrt{\frac{q(x)}{p(x)}}\cdot p(x), \end{aligned}$$

i.e.,

$$\sum_{x\in\mathfrak{X}} \frac{p^2(x)}{q(x)} + rR\sum_{x\in\mathfrak{X}} q(x) \leq (r+R)\sum_{x\in\mathfrak{X}} p(x).$$

In addition, as

$$\sum_{x\in\mathfrak{X}} p(x) = \sum_{x\in\mathfrak{X}} q(x) = 1,$$

we obtain

$$\sum_{x\in\mathfrak{X}} \frac{p^2(x)}{q(x)} \leq r + R - rR,$$

that is, the first inequality in (2.10).

The second inequality in (2.10) is obvious by the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2,\ a,b\in\mathbb{R}.$$

Finally, the case of equality follows by the similar case in the Diaz-Metcalf result. □

The following theorem holds.

Theorem 6. *Let p,q,r,R be as in Lemma 1. Then we have the inequality*

$$0 \leq KL(p,q) \leq \log[(1-r)(R-1)+1] \leq \log\left[\frac{1}{4}(R-r)^2+1\right].$$

Equality holds if and only if $p(x)=q(x)$ for all $x\in\mathfrak{X}$.

The proof is obvious by Theorem 4 and Lemma 2 and we omit the details.

The following corollary holds.

Corollary 2. *Let p,q,r,R be as in Lemma 1. If $\varepsilon>0$ and*

$$0 < R - r < 2\sqrt{e^{\varepsilon}-1},$$

then we have the inequality

$$KL(p,q) \leq \varepsilon.$$

3. Applications for Shannon's Entropy

The *entropy* of a random variable is a measure of the uncertainty of the random variable; it is a measure of the amount of information required on the average to describe the random variable.

Let $p(x), x\in\mathfrak{X}$ be a probability mass function. Define the *Shannon's entropy* of a random variable X having the probability distribution p by

$$H(X) := \sum_{x\in\mathfrak{X}} p(x)\log\frac{1}{p(x)}. \tag{3.1}$$

In the above definition, we use the convention (based on continuity arguments) that $0\log\left(\frac{0}{q}\right)=0$ and $p\log\left(\frac{p}{0}\right)=\infty$.

Now, assume that $|\mathfrak{X}|$ ($card\,(\mathfrak{X})=|\mathfrak{X}|$) is finite and let $u(x)=\frac{1}{|\mathfrak{X}|}$ be the uniform probability mass function on $\mathfrak{X}$. It is well known that [4, p. 27]

$$KL(p\,\|\,u)=\sum_{x\in\mathfrak{X}}p(x)\log\left(\frac{p(x)}{u(x)}\right)=\log|\mathfrak{X}|-H(X). \tag{3.2}$$

The following result is important in Information Theory [4, p. 27].

Theorem 7. *Let X,p and $\mathfrak{X}$ be as above. Then*

$$H(X)\le\log|\mathfrak{X}|, \tag{3.3}$$

with equality if and only if X has a uniform distribution over $\mathfrak{X}$.

Using some of the results obtained in Section 2 for Kullback-Leibler distances and χ^2-distances, we now develop some new inequalities for Shannon's entropy.

Theorem 8. *Let X,p and $\mathfrak{X}$ be as above. Then*

$$0\le\log|\mathfrak{X}|-H(X)\le\log\left(|\mathfrak{X}|\sum_{x\in\mathfrak{X}}p^2(x)\right). \tag{3.4}$$

Equality holds if and only if $p(x)=\frac{1}{|\mathfrak{X}|}$ for all $x\in\mathfrak{X}$.

Proof. The proof follows by the inequality (2.1), choosing $q=u$ and taking into account that

$$KL(p,u)=\log|\mathfrak{X}|-H(X);$$
$$D_{\chi^2}(p,u)=|\mathfrak{X}|\sum_{x\in\mathfrak{X}}p^2(x)-1.$$

We omit the details. □

Remark 2. *The second inequality in (3.4) is equivalent to*

$$H(X)+\log\left(\sum_{x\in\mathfrak{X}}p^2(x)\right)\ge 0. \tag{3.5}$$

If we denote by $E(X)$ (informational energy of X) the sum $\sum_{x\in\mathfrak{X}}p^2(x)\le 1$, then, from (3.5), we obtain

$$H(X)\ge\log\frac{1}{E(X)}\ge 0.$$

An equivalent inequality is

$$E(X)\ge\exp[-H(X)]>0.$$

The following upper bound for the informational energy E also holds.

Theorem 9. *Let X,p and $\mathfrak{X}$ be as above. Then*

$$E(X)\le\frac{1}{|\mathfrak{X}|}\frac{(P+p)^2}{4pP}, \tag{3.6}$$

provided that $p=\min_{x\in\mathfrak{X}}p(x)$, $P=\max_{x\in\mathfrak{X}}p(x)$. Equality holds in (3.6) if and only if $p(x)=\frac{1}{|\mathfrak{X}|}$ for all $x\in\mathfrak{X}$.

The proof follows from Lemma 1 setting $q = u$.

Another result concerning an upper bound for the difference $\log|\mathfrak{X}| - H(X)$ is embodied in the following theorem.

Theorem 10. *Let X, p and $\mathfrak{X}$ be as in Theorem 9. Then we have*

$$0 \le \log|\mathfrak{X}| - H(X) \le \log\left[\frac{(P-p)^2}{4pP} + 1\right]. \tag{3.7}$$

Equality holds in (3.7) if and only if $p(x) = \frac{1}{|\mathfrak{X}|}$ for all $x \in \mathfrak{X}$.

The proof is obvious by Theorem 5 by choosing $q = u$. We omit the details.

Corollary 3. *Under the assumptions of Theorem 10 and assuming that*

$$\rho := \frac{P}{p}\,(\ge 1)$$

and that ρ satisfies the inequality

$$\rho \le 2e^{\varepsilon} - 1 + 2\sqrt{e^{\varepsilon}(e^{\varepsilon} - 1)}$$

for a given $\varepsilon > 0$. Then we have the estimate

$$0 \le \log|\mathfrak{X}| - H(X) \le \varepsilon.$$

Using Lemma 2, we can state another upper bound for the informational energy $E(X)$.

Theorem 11. *Let X, p and $\mathfrak{X}$ be as in Theorem 9. Then we have the inequality:*

$$\begin{aligned} E(X) &\le \frac{1}{|\mathfrak{X}|} + |\mathfrak{X}|\left(\frac{1}{|\mathfrak{X}|} - p\right)\left(P - \frac{1}{|\mathfrak{X}|}\right) \\ &\le \frac{1}{|\mathfrak{X}|} + \frac{1}{4}|\mathfrak{X}|(P-p)^2. \end{aligned}$$

Equality holds if and only if $p(x) = \frac{1}{|\mathfrak{X}|}$ for all $x \in \mathfrak{X}$.

The proof follows by Lemma 2 for $q = u$. We omit the details.

Finally, we have the following upper bound for $\log|\mathfrak{X}| - H(X)$.

Theorem 12. *Under the assumptions of Theorem 9, we have the inequality*

$$\begin{aligned} 0 &\le \log|\mathfrak{X}| - H(X) \le \log\left[1 + |\mathfrak{X}|^2\left(\frac{1}{|\mathfrak{X}|} - p\right)\left(P - \frac{1}{|\mathfrak{X}|}\right)\right] \\ &\le \log\left[1 + \frac{1}{4}|\mathfrak{X}|^2(P-p)^2\right]. \end{aligned}$$

Equality holds if and only if $p(x) = \frac{1}{|\mathfrak{X}|}$ for all $x \in \mathfrak{X}$.

The proof follows by Theorem 6 for $q = u$.

Corollary 4. *If, for a given $\varepsilon > 0$, we have*

$$0 < P - p < \frac{2}{|\mathfrak{X}|}\sqrt{e^{\varepsilon} - 1},$$

then

$$0 \le \log|\mathfrak{X}| - H(X) \le \varepsilon.$$

4. Applications for the Mutual Information

We consider the *mutual information,* which is a measure of the amount of information that one random variable contains about another random variable. It is the reduction of uncertainty of one random variable due to the knowledge of the other [4, p. 18].

To be more precise, consider two random variables X and Y with a joint probability mass $r(x,y)$ and marginal probability mass functions $p(x)$ and $q(y)$, $x \in \mathfrak{X}$, $y \in \mathcal{Y}$. The mutual information is the relative entropy between the joint distribution and the product distribution. That is,

$$\begin{aligned} I(X;Y) &= \sum_{x\in\mathfrak{X}, y\in\mathcal{Y}} r(x,y)\log\left[\frac{r(x,y)}{p(x)q(y)}\right] \\ &= D(r \,\|\, pq). \end{aligned}$$

in following theorems equality holds if and only if X and Y are independent.

The following result is well known [4, p. 27].

Theorem 13. *(Non-negativity of mutual information) For any two random variables* X, Y

$$I(X;Y) \geq 0. \tag{4.1}$$

The following counterpart of (4.1) holds.

Theorem 14. *For any two random variables* X, Y *we have*

$$0 \leq I(X;Y) \leq \log\left[\sum_{(x,y)\in\mathfrak{X}\times\mathcal{Y}} \frac{r^2(x,y)}{p(x)q(y)}\right]. \tag{4.2}$$

Proof. Follows by the inequality (2.1), taking into account that

$$KL(r, pq) = I(X;Y),$$

and

$$D_{\chi^2}(r, pq) = \sum_{(x,y)\in\mathfrak{X}\times\mathcal{Y}} \frac{r^2(x,y)}{p(x)q(y)} - 1.$$

We omit the details. □

Now we consider the following "mutual information" associated with the χ^2-distance function. Therefore, for two random variables X and Y as above, consider the χ^2-*mutual information* given by

$$I_{\chi^2}(X,Y) = D_{\chi^2}(r, pq) = \sum_{(x,y)\in\mathfrak{X}\times\mathcal{Y}} \frac{r^2(x,y)}{p(x)q(y)} - 1. \tag{4.3}$$

It is obvious that $I_{\chi^2}(X,Y) \geq 0$ and, by (4.2),

$$\begin{aligned} 0 &\leq I(X;Y) \leq \log\left[I_{\chi^2}(X,Y) + 1\right] \\ &\leq I_{\chi^2}(X,Y). \end{aligned}$$

We now point out an upper bound for the χ^2-mutual information.

Theorem 15. *Let X and Y be as above. Suppose that*

$$0 < t \le \frac{r(x,y)}{p(x)\,q(y)} \le T \quad \text{for all } (x,y) \in \mathfrak{X} \times \mathcal{Y}.$$

Then we have the inequality

$$I_{\chi^2}(X,Y) \le \frac{(T-t)^2}{4tT}.$$

The proof follows by Lemma 1 and we omit the details.

Using Theorem 5, we can state the following upper bound for the usual mutual information $I(X,Y)$.

Theorem 16. *Let X and Y be as in Theorem 15. Then we have the inequality*

$$0 \le I(X;Y) \le \log\left[\frac{(T-t)^2}{4tT} + 1\right].$$

Equality holds if and only if X and Y are independent.

The following corollary also holds.

Corollary 5. *Let X and Y be as in Theorem 15. If*

$$\rho := \frac{T}{t} \ (\ge 1)$$

satisfies the inequality

$$\rho \le 2e^{\varepsilon} - 1 + 2\sqrt{e^{\varepsilon}(e^{\varepsilon}-1)},$$

then we have the inequality

$$0 \le I(X;Y) \le \varepsilon.$$

Another bound for χ^2−mutual information is embodied in the following theorem.

Theorem 17. *Let X and Y be as in Theorem 15. Then we have the inequality*

$$I_{\chi^2}(X,Y) \le (1-t)(T-1) \le \frac{1}{4}(T-t)^2.$$

The proof is obvious by Lemma 2 and we omit the details.

Lastly, by the use of Theorem 6, we have

Theorem 18. *Under the assumptions of Theorem 15 for X and Y, we have the bound:*

$$0 \le I(X;Y) \le \log[(1-t)(T-1)+1] \le \log\left[\frac{1}{4}(R-r)^2 + 1\right].$$

Corollary 6. *For a given $\varepsilon > 0$, if*

$$0 \le T - t \le 2\sqrt{e^{\varepsilon}-1},$$

then we have the inequality

$$0 \le I(X;Y) \le \varepsilon.$$

References

[1] S. KULLBACK and R.A. LEIBLER, On information and sufficiency, *Annals Maths. Statist.,* **22** (1951), 79 - 86.

[2] S. KULLBACK, *Information Theory and Statistics,* J. Wiley, New York, 1959.

[3] A. BEN-TAL, A. BEN-ISRAEL and M. TEBOULLE, Certainty equivalents and information measures: duality and extremal, *J. Math. Anal. Appl.,* **157** (1991), 211 - 236.

[4] T.M. COVER and J.A. THOMAS, *Elements of Information Theory*, John Wiley & Sons, Inc., 1991.

[5] S.S. DRAGOMIR and N.M. DRAGOMIR, An inequality for logarithms and its application in coding theory, *Indian J. of Math.* (in press).

[6] S.S. DRAGOMIR, M. SCHOLZ and J. SUNDE, Some upper bounds for relative entropy and applications, *Computers and Mathematics Applications.*Vol. 39 (2000), No. 9-10, 91-100.

[7] D.S. MITRINOVIĆ, *Analytic Inequalities,* Springer Verlag, 1970.

[8] M. MATIĆ, *Jensen's Inequality and Applications in Information Theory (* in Croatian), Ph.D. Dissertation, Split, Croatia.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC 8001, Australia

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

Royal Australian Air Force, ARDU, PoBox 1500, Salisbury SA 5108,

E-mail address: vgluscev@spam.adelaide.edu.au