TORRICELLI'S PROBLEM IN THE MINKOWSKIAN PLANE

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Abstract. We analyse the problem to minimize the sum of distances of a point to three fixed points in a plane, which is endowed with an indefinite inner product structure. The Minkowskian plane is prefered since it allows an immediate physical interpretation. 46C20; 83A05 Torricelli's point; Minkowskian space-time

1. Introduction

It is well known that the Torricelli's point of a triangle in the Euclidean plane is either the vertex of the obtuse angle if there is one, or the unique point wherefrom each side is seen under $\frac{2\pi}{3}$, if the triangle has only acute angles. In [3] it is shown that the situation is similar in arbitrary definite inner product spaces, and also in normed spaces (as future work in the same paper is announced).

Our purpose is to go back to the Torricelli's problem in a plane, but considering this plane organized by a nondegenerate indefinite inner product (terminology will be adopted from [2] and [1]). More exactly, on $M = \mathbb{R}^2$ we consider the inner product $(.,.) : M \times M \longrightarrow \mathbb{R}$ of values

$$((t_1, x_1), (t_2, x_2)) = t_1 t_2 - x_1 x_2 \tag{*}$$

i.e. M is organized as a Minkowskian plane of events $e = (t, x)$ with units scaled such that the speed of light becomes $c = 1$. The first specific feature of such a space is the possibility of distinguishing *positive vectors (time-like events)*, for which $(e, e) > 0$, from *negative vectors (space-like events)*, when $(e, e) < 0$, and from *neutral vectors (signal, or light-like events),* which have null square (without being necessarily null) i.e. $(e, e) = 0$.

This classification of the vectors leads to different types of triangles, but first of all we have to mention that because the notion of *angle* makes sense only for pairs of vectors of the same kind, it's more appropriate to speak of *trievents* instead of *triangles.* According to the nature of its sides, a trievent (e_1, e_2, e_3) can be of type:

- *time-time-time* (briefly *t.t.t.*) if all of $e_2 - e_1, e_3 - e_2, e_3 - e_1$ are positive;
- *time-light-time (t.l.t.)* if $e_2 - e_1$ and $e_3 - e_1$ are positive, but $e_3 - e_2$ is neutral;
- *time-space-time (t.s.t.)* if $e_2 - e_1$ and $e_3 - e_1$ are positive, but $e_3 - e_2$ is negative;

etc.

As usually, the inner product (*) produces a *geometry* on M, which could be rather called *chronometry*, or better *geochronometry,* since it equally involves measurements of either space and time. Passing from a definite inner product to an indefinite one turns out to be more profound than repeating facts and techniques in other terms and modified formulas, i.e. we have to rebuild the specific foundation consisting of basic properties as well as practical significance. In the particular case of M, the most useful mathematical tools derived from $(.,.)$ are:

- The relation of *orthogonality* $\perp$, defined by
$$e_1 \perp e_2 \Longleftrightarrow (e_1, e_2) = 0$$
- The *causal* relation

$$K = \{((t_1, x_1), (t_2, x_2)) \in M^2 : t_2 - t_1 > |x_2 - x_1|\} \cup \delta,$$

where δ is the diagonal of M^2 (i.e. *equality* on M).

- The *temporal* norm $\|\cdot\|_+ : K[0] \longrightarrow \mathbb{R}^+$,of values

$$\|(t,x)\|_+ = \sqrt{t^2 - x^2}$$

- The *hyperbolic* angle between two vectors in the "future cone" $K[0]$, measured by the formula

$$ch\alpha = \frac{(e_1, e_2)}{\|e_1\|_+ \|e_2\|_+}.$$

¿From the physical point of view (see [4],[2],[1], etc.) orthogonality is interpreted in terms of *simultaneity* of $\pm e_1$ relative to an inertial observer defined by e_2. Causality expresses the possibility of transmitting information from e_1 to e_2 using a particle / observer, and $\|e_2 - e_1\|_+$ measures the *proper time* of a particle that carries this information. Finally, hyperbolic angles are useful to describe the *relative speed* of two observers, since $th\alpha = v/c$.

Of course, the above list is far from being complete; there should be added a lot of notions, especially the *duals*, as for example:

a) The *spatial* relation

$$S = \{((t_1, x_1), (t_2, x_2)) \in M^2 : |t_2 - t_1| < |x_2 - x_1|\} \cup \delta,$$

containing pairs of causally noninfluenced events, and

b) the *spatial* norm $\|\cdot\|_- : S[0] \longrightarrow \mathbb{R}^+$, which has the formula

$$\|(t,x)\|_- = \sqrt{x^2 - t^2},$$

and measures *proper space.*

Apart from their physical meaning, these notions satisfy the specific mathematical requirements. In particular K is a linear order on M, and $\|\cdot\|_+$ is a *super-additive norm* (briefly S.a.), i.e. the following conditions hold:

1. $\|e\|_+ = 0 \Longleftrightarrow e = 0$ (nondegeneracy)
2. $\|\lambda e\|_+ = \lambda \|e\|_+$ for all $\lambda \in \mathbb{R}^+$ and $e \in K[0]$ (homogeneity)
3. $\|e_1 + e_2\|_+ \geq \|e_1\|_+ + \|e_2\|_+$ whenever $e_1, e_2 \in K[0]$ (S.a.).

In addition, some standard schemes are used to derive new notions (see [1], [5], etc.). In this note we need to deal at least with:

- The *causal interval* (briefly K-interval) of end events e_1 and e_2:
$$[e_1, e_2]_K = \{e \in M : (e_1, e) \& (e, e_2) \in K\}.$$
- The *closure* of K, noted

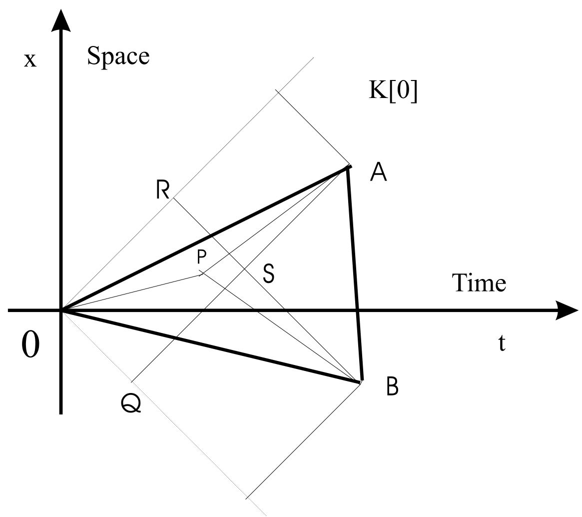

FIGURE 1

$$\overline{K} = \{((t_1, x_1), (t_2, x_2)) \in M^2 : t_2 - t_1 \geq |x_2 - x_1|\},$$

on which we extend $\|\cdot\|_+$ by null values.

- The *temporal* metric $d_+ : K \longrightarrow \mathbb{R}_+$, derived from $\|\cdot\|_+$ as

$$d_+(e_1, e_2) = \|e_2 - e_1\|_+$$

etc.

Generally speaking, each problem concerning trievents relative to the structure generated by (*), should be analyzed for all the possible types. Fortunately, in practice this si not the case because:

a) Some combinations defining the type are impossible, e.g. $l.l.l.$

b) There exist pairs in duality, which are treated similarly, e.g. $t.t.t.$ and $s.s.s.$

c) Light-like sides can be considered extreme positions of either time-like and space-like vectors.

For this reason we limit ourselves to analyze only the $t.s.t.$ and $t.t.t.$ types.

2. Torricelli's problem for t.s.t. trievents

To get a simple setting of the problem, let us consider the trievent ΔOAB of type $t.s.t.$ as in Fig.1, that is $e_1 = O = (0, 0)$, $e_2 = A = (t_1, x_1)$, and $e_3 = B = = (t_2, x_2)$, such that $t_1 > x_1$, $t_2 > x_2$, but $|t_2 - t_1| < |x_2 - x_1|$. For shortness we will note OA for $\|e_1\|_+ = d_+(O, A)$, etc.

The problem is to identify the positions of $e = P = (t, x)$ that minimizes the Torricellian functional $\mathcal{T}$, defined by

$$\mathcal{T}(P) = \|e\|_+ + d_+(P, A) + d_+(P, B) \overset{not.}{=} OP + PA + PB =$$
$$= \sqrt{t^2 - x^2} + \sqrt{(t_1 - t)^2 - (x_1 - x)^2} + \sqrt{(t_2 - t)^2 - (x_2 - x)^2}$$

Because $\mathcal{T}$ is expressed in terms of d_+ , point P (= event e) is restricted to the interior of the rectangle $ORSQ = [O, S]_{\overline{K}}$. The Torricelli's problem makes sense since based on super-additivity we see that $\mathcal{T}(P) < \mathcal{T}\ (O)$ whenever $P \neq O$.

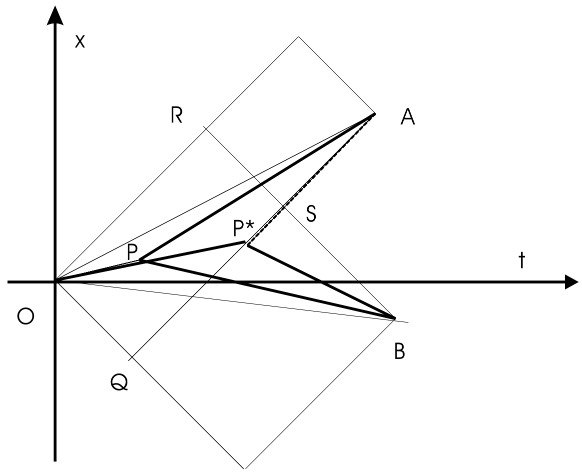

FIGURE 2

The analytic method involving stationary points, i.e. null differential, appears as very fruitful in definite inner product settings, including [3], but doesn't work for $\mathcal{T}$ from above. In fact, a simple algebraic manipulation of $\frac{\partial \mathcal{T}}{\partial t} = 0$ and $\frac{\partial \mathcal{T}}{\partial x} = 0$ (like addition of some squares) leads to an impossible value of ch, namely

$$\frac{(e_1 - e, e_2 - e)}{\|e_1 - e\|_+ \|e_2 - e\|_+} = ch\alpha = -\frac{1}{2}.$$

It is easy to see that the similar result, $\cos \alpha = -\frac{1}{2}$, offers the solution in the case of an Euclidean structure. The failure of this technique in the Minkowskian plane is explained by the nondifferentiability of $\mathcal{T}$ at stationary points, but it is useful as a hint to look for the minimum of $\mathcal{T}$ where the square roots are not differentiable, i.e. on the sides of $[O, S]_{\overline{K}}$. On the other hand it is clear that we have to try another method for solving the problem, and what we propose here is to reduce it to simpler properties obtained by direct / synthetic reasons.

Lemma 1. *For each $P \in [O, S]_K$ there exists $P^* \in [R, S] \cup [QS]$ such that*

$$\mathcal{T}\ (P^*) \leq \mathcal{T}\ (P)\ .$$

Proof. Let P^* be the intersection of the half-line $[OP$ with $[R, S] \cup [QS]$, as in fig.2.

Adding the obvious relations $PA > P^*A = 0$, $PA \geq PP^* + P^*B$ (S.a.), and $OP = OP$, and taking into account that $OP + PP^* = OP^*$, we immediately obtain $\mathcal{T}(P) \geq \mathcal{T}(P^*)$. ∎

Lemma 2. *If $\triangle OPA$ and $\triangle ORA$ are t.t.t. trievents such that the interior of $\triangle ORA$ contains P, then the following extension of the super-additivity holds*

$$OP + PA \geq OR + RA,$$

i.e. the more roundabout a broken evolution is, the shorter proper time it has.

Proof. Let us note $V = [OP \cap [RA]$, like in fig.3.

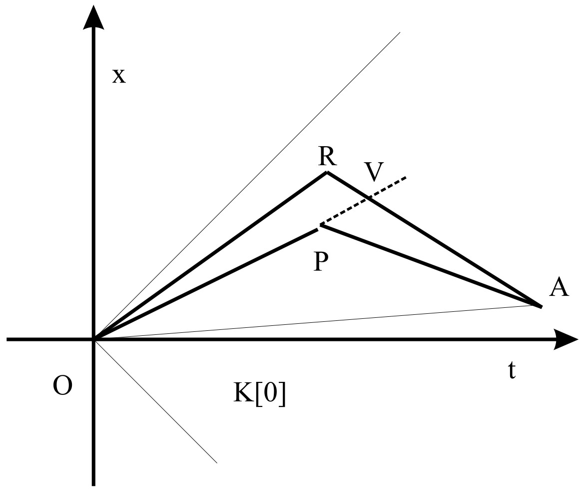

FIGURE 3

According to S.a., the inequalities $OV \geq OR + RV$, and $PA \geq PV + VA$ hold in the trievents $\triangle ORV$ and $\triangle PVA$. The desired relation results by adding the first inequality to the reversed second one. ∎

Now we can formulate and prove the main result:

Proposition 3. *If $\triangle OAB$ is a t.s.t trievent with $O = (0,0)$, $A = (t_1, x_1)$, $B = (t_2, x_2)$, and*

$$R = (\tfrac{1}{2}(t_2 + x_2), \tfrac{1}{2}(t_2 + x_2))$$
$$S = (\tfrac{1}{2}(t_1 + t_2 - x_1 + x_2), \tfrac{1}{2}(x_1 + x_2 - t_1 + t_2))$$
$$Q = (\tfrac{1}{2}(t_1 - x_1), \tfrac{1}{2}(x_1 - t_1)).$$

as in fig.4, then

$$\inf_{P \in [O,S]_{\overline{K}}} \mathcal{T}(P) = \min\{RA, OS, QB\}.$$

Consequently at least one element of $\{R, S, Q\}$ is a Torricellian point / event.

Proof. According to lemma 1 we have to look for Torricellian points only on the sides $[RS] \cup [QS]$. If we consider $A' = [OA] \cap [RS]$ and $B' = [OB] \cap [QS]$, then several positions of P are possible. For example, if $P \in [RA']$,then using Lemma 2, we obtain $OR + RA \leq OP + PA$. Because $RB = PB = 0$,this also means that $\mathcal{T}(R) \leq \mathcal{T}(P)$. Similarly, $\mathcal{T}(S) \leq \mathcal{T}(P)$ holds for $P \in [A'S]$, and so on. In other words, $d_+(R, A) = RA$, $d_+(O, S) = OS$, and $d_+(Q, B) = QB$ are local minimal values of the Torricellian functional $\mathcal{T}$.

Finally we may conclude that there is a single Torricellian point, say R, if $RA < OS$ and $RA < QB$; similarly this point can be S or Q. There are two Torricellian points, say R and S,if $RA = OS < QB$, etc. If $RA = OS = QB$, the three points R, S, Q are all Torricellian. ∎

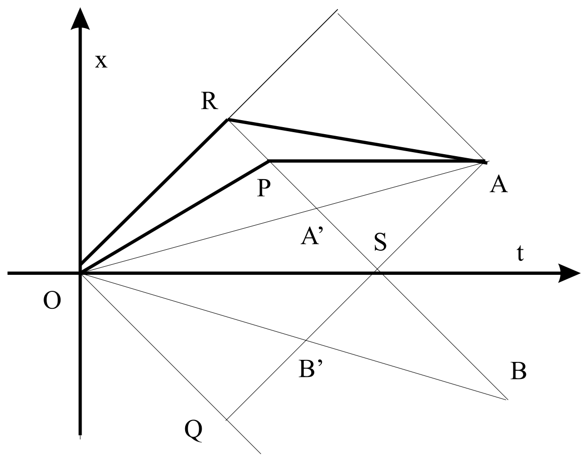

FIGURE 4

The physical interpretation of the result can be formulated in terms of minimal proper time in the process of transmitting information from O to A and B. The practical solution is to locate some devices at R, S, or Q , which can receive the message of O and split it to be send to A and B using a signal or a particle. The location is the solution of the corresponding Torricelli's problem.

3. TORRICELLI'S PROBLEM FOR T.T.T. TRIEVENTS

Let $\triangle OAB$ be a trievent of t.t.t. type, where $O = (0, 0)$, $A = e_1 = (t_1, x_1)$, and $B = e_2 = (t_2, x_2)$, such that $e_1 \in K[O]$ and $e_2 \in K[e_1]$. More exactly, this means that $t_1 > |x_1| > 0$ and $t_2 - t_1 > |x_2 - x_1|$, like in fig.5; an immediate consequence is $e_2 \in K[O]$. The problem to minimize (the same) Torricellian functional $\mathcal{T}$ makes sense on the $\overline{K}$-interval $[O, A]_{\overline{K}}$. The solution is contained in the following:

Proposition 4. *If $\triangle OAB$ is the t.t.t. trievent from above, then (see fig.5)*

$$\inf_{P \in [O,A]_{\overline{K}}} \mathcal{T}(P) = \min\{SB, VB\}$$

where $S = (\frac{1}{2}(x_1 + t_1), \frac{1}{2}(x_1 + t_1))$ and $V = (\frac{1}{2}(x_1 - t_1), \frac{1}{2}(x_1 - t_1))$. At least one member of $\{S, V\}$ is a Torricellian point / event.

Proof. Because line OB divides the $\overline{K}-$interval $[O, A]_{\overline{K}}$ in two regions, we have to

consider two cases, namely P is interior to the polygonal line $OSAC$, or P is interior to $\triangle OCV$. In the first case, according to Lemma 2, we have

$$OS + SB \leq OP + PB.$$

Adding here the obvious inequality $0 = SA \leq PA$, it follows that

$$SB = d_+(S, B) = \mathcal{T}(S) \leq \mathcal{T}(P).$$

Similarly, in the second case we have

$$VB = d_+(V, B) = \mathcal{T}(V) \leq \mathcal{T}(P).$$

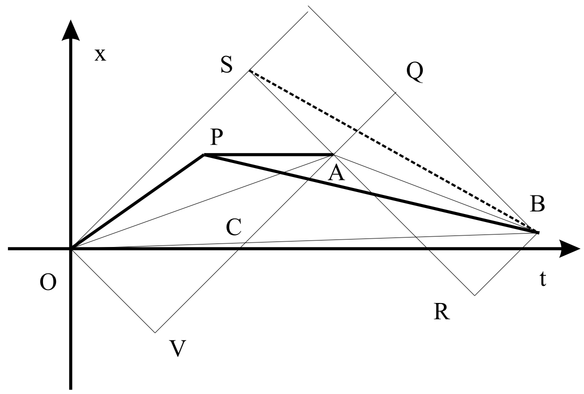

FIGURE 5

Consequently the sought for infimum is attained either at S or at V; more exactly, S is the Torricelli's point if $SB < VB$, and V is the point if the converse inequality holds, but it's possible both S and V be Torricellian if $SB = VB$. ∎

Because the form of the functional $\mathcal{T}$ coincides to that considered for t.s.t. trievents, the physical meaning of the problem and its solution will be the same. The situation changes if the Torricelli's functional has another form, e.g.

$$\widetilde{\mathcal{T}}(P) = OP + AP + PB = d_+(O,P) + d_+(A,P) + d_+(P,B)$$

This time the problem makes sense for $P \in [A,B]_{\overline{K}}$, and the solution differs from the previous one even if $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ have identical expressions in variables (t,x).

Proposition 5. *If $\triangle OAB$ is a t.t.t. trievent and $P \in [A,B]_{\overline{K}}$, like in fig.5, then*

$$\inf_{P\in[A,B]_{\overline{K}}} \widetilde{\mathcal{T}}(P) = \min\{OQ, OR\}$$

and at least one of Q and R is a Torricellian point / event.

Proof. We may reason as for the previous proposition. ∎

Condition $P \in [A,B]_{\overline{K}}$ can be physically interpreted as the reception in P of some messages from both O and A, followed by the emission of a combined message to B. The problem requires to do this in the shortest possible proper time, and the solution indicates the transit through the Torricelli's point / event.

Because the other types of trievents can be treated in essentially similar ways, we may admit that the propositions from above completely solve the Torricelli's problem in the Minkowskian space-time. Finally we remark that transforming the Minkowskian plane into an Euclidean one via the "complexification" $x_0 = ict$ fails to furnish the correct solution of the problem in indefinite inner product settings.

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