A NOTE ON THE ASYMPTOTIC STABILITY

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Abstract. In this paper, we study the influence of the perturbing term in equation x'=f(t,x)+g(t,x), on the asymptotically behaviour of x'=f(t,x).

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1. **Introduction.**

Consider the following systems of differential equations:

$$x' = f(t,x), \tag{1}$$

$$x' = f(t,x) + g(t,x), \tag{2}$$

where f and g are n-vectors, continuous on the domain D={(t,x): t∈I, x∈ Ω} with I:=[0,+∞) and an open set of $\mathbf{R}^n$. Here x(t) and x(t;t_0,x_0) are solutions of systems (1) and (2), such that x(t_0;t_0,x_0)=x_0.

Thorough this paper C(I), CI(I), CP(I) and CC(I) denote the families of continuous function, continuous increasing functions, continuous and positive functions and continuous functions satisfying uf(u)>0 for u≠ 0 with f in that class, respectively, on the real interval I. We also consider the following functional class (cf. [21]):

L=$\big\{\lambda(t) \in C(I) : \lambda(t) \geq 0, \int_0^\infty \lambda(t)dt < \infty\big\}$,

P=$\big\{p(t) \in C(I) : p(t) \geq 0, \int_0^\infty p(t)dt < \infty\big\}$,

F=$\Big\{\phi(u) \in C(R) : \phi(u) > 0, \text{nodecreasing and } \int_0^\infty \frac{du}{\phi(u)} = \infty\Big\}$.

There are many results concerned with relationships between solutions of an unperturbed system and solutions of its perturbed system, especially concerning the stability of solutions of the perturbed system (we refer to reader to the bibliography and references cited therein). The purpose of this paper is to study the influence of the perturbing term g(t,x) in (2) on the asymptotic properties of (1) under non-usual assumptions, in particular assumptions of type that g is small in some sense, are not considered here.

The main tool used in this work is the Second Method of Lyapunov. Foundations of stability theory for ordinary differential equations can be found, for example, in Yoshizawa [34] and Demidovich [6].

Denote by I_x the maximal interval in which the solution x(t) of (1) exists. For any solution x(t) of (1) we have either $I_x = (0, t^+), 0 < t^+ \leq \infty$, or $I_x = (t^-, t^+), 0 \leq t^- \leq t^+ \leq \infty$. In system (1), let f(t,x) be continuous on D. We say that a solution x(t) of (1) is defined in the future (or continuable) if $t^+ = \infty$.

For a constant $\alpha > 0$ and any convenient norm, let $S_\alpha = \{x : |x| < \alpha\}$.

First of all, the definitions of boundedness of solutions will be given (cf. [33]). Suppose that through every point of D passes only one solution of (1).

A solution x(t) of (1) is bounded, if there exists a $\beta > 0$ such that $x \in S_\beta$ for all $t \geq t_0$, where β may depend on each solution.

The solutions of (1) are equi-bounded, if for any $\alpha > 0$ and $t \in I$, there exists a $\beta = \beta(t_0, \alpha) > 0$ such that if $x_0 \in S_\alpha$, $x(t) \in S_\beta$.

The solutions of (1) are ultimately bounded for bound B, if there exists a $\beta > 0$ and a T>0 such that for every solution $x(t)$, $x(t) \in S_\beta$ for all $t \geq t_0 + T$, where B is independent of the particular solution while T may depend on each solution.

The solutions of (1) are equi-ultimately bounded for bound B, if there exists a B>0 and if corresponding to any $\alpha > 0$ and $t_0 \in I$, there exists a $T = T(t_0, \alpha) > 0$ such that $x_0 \in S_\alpha$ implies that $x(t) \in S_\beta$ for all $t \geq t_0 + T$.

Let x(t) be a solution of (1) defined in the future.

We say that x(t) is stable (in the sense of Lyapunov) if given $\varepsilon > 0$, $t_0 \in I_x$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $x_0 \in S_\delta$ implies that x(t) exists in the future and $x \in S_\varepsilon$ for all $t \geq t_0$.

We say that x(t) is asymptotically stable if it is stable and there exists a $\delta = \delta(t_0) > 0$ such that $\|x\| \longrightarrow 0$ for all x such that $x_0 \in S_\delta$.

We say that x(t) is asymptotically stable in the whole if it is stable and every solution of (1) tends to zero as $t \longrightarrow \infty$.

We say that x(t) is quasi-equiasymptotically stable if given $\varepsilon > 0$, $t_0 \in I_x$, there exists $\delta = \delta(t_0) > 0$ and a $T(t_0, \varepsilon) > 0$ such that if $x_0 \in S_\delta$, $x \in S_\varepsilon$ for all $t \geq t_0 + T(t_0, \varepsilon)$.

We say that x(t) is equiasymptotically stable if it is stable and is quasi- equiasymptotically stable.

We say that x(t) is quasi-equiasymptotically stable in the whole if for any $\alpha > 0$, any $\varepsilon > 0$ and any $t_0 \in I_x$, there exists $T(t_0, \alpha, \varepsilon) > 0$ such that if$x \in S_\delta$, $x \in S_\varepsilon$ for all$t \geq t_0 + T(t_0, \alpha, \varepsilon)$.

We say that x(t) is equiasymptotically stable in the whole if it is stable and quasi-equiasymptotically stable in the whole.

Here, it is noticed that if $x(t) \equiv 0$ is the unique solution of (1) through (0,0) quasi-equiasymptotically stability implies the stability of $x(t) \equiv 0$ (see [34]).

Other definitions of different types of stability and boundedness we can found in [2] and [34].

2. **Results.**

Suppose that there exists a Lyapunov function V(t,x) defined on IxS_α, satisfying the conditions:

$$A(t)\varphi\left(\|x\|\right) \leq V(t,x) \leq B(t)\varphi\left(\|x\|\right), \tag{3}$$

$$V'_{(1)} \leq -C(t)\varphi\left(\|x\|\right), \tag{4}$$

$$\|gradV.g\| \leq \gamma(t)\varphi\left(\|x\|\right), \tag{5}$$

where A, B y C are positive continuous functions defined on I, such that:

$$A(t) \geq a \geq 1 \; and \; \frac{C(t)}{B(t)} \geq b, \tag{6}$$

for some positive constants a, b and where φ is a positive and continuous increasing function on R such that

(7) $$\int_0^t \frac{dr}{\varphi(r)} = \infty \ as \ t \longrightarrow \infty.$$

Under assumptions (3) y (4), we can conclude the equi-boundedness of solutions of (19) (see Theorem 10.1 of [34] or Lemma 1 of [21]), by other hand, using a simple variant of Theorem 10.5 of [34], the solutions of (1) are equi-ultimatelly bounded if (3) and (4) hold.

Here we will give some results on these properties of solutions of the perturbed system (2) under condition (7) on the perturbing term g(t,x) in which

(8) $$\gamma = \limsup_{t\longrightarrow+\infty} \left\{ \frac{1}{t}\int_0^t \gamma(r)dr \right\} < ab,$$

with γ sufficiently small. We choose the solution $x(t;t_0,x_0)$ of the system (2) which satisfies that:

(9) $$x(0) = x_0, x_0 \in S_a.$$

On this solution, the derivative $V'_{(2)}(t,x)$ satisfies:

$$V'_{(2)}(t,x) = \frac{\partial V}{\partial t} + (f+g)gradV \leq -C(t)\varphi\left(\|x\|\right) + \gamma(t)\varphi\left(\|x\|\right)$$

and from the condition (5) we have:

(10) $$V'_{(2)}(t,x) = -\frac{C(t)}{D(t)}V(t,x) + \frac{\gamma(t)}{A(t)}V(t,x).$$

It follows then, by using condition (6), that:

(11) $$V'_{(2)}(t,x) \leq \left[\frac{\gamma(t)}{a} - b\right] V(t,x).$$

Taking in account (3), and above result, we have that solutions of system (2) are equi-bounded in virtue of Theorem 10.1 of [34].

Since V(t,x) is continuous, there exists a $K(t_0,a) > 0$ such that if $x \in S_a$, $V(t_0,x_0) \leq K(t_0,a)$. Let $x(t;t_0,x_0)$ be a solution of (2) such that $x_0 \in S_a$. Suppose that for $t > t_0 + 1/c \ log[K(t_0,a)/\varphi(B)]$, $c = (b - \gamma/a)$, $V(t,x(t;t_0,x_0)) \geq \varphi(B)$, i.e. $x(t;t_0,x_0) \in S_B^c$, which means the not ultimately boundedness of the solutions of (2).

Putting $\lambda(t) = \frac{\gamma(t)}{a} - b \in L(I)$ in (11), we have by (8) that $L(t) \geq c(t-t_0)$, where$L(t) = \int_{t_0}^t \lambda(s)ds$. So, we obtain

$$\begin{aligned}\varphi(B) &\leq V(t,x(t;t_0,x_0)) < V(t_0,x_0)exp\{-c(t-t_0)\} < \\ &< K(t_0,a)exp\{-logK(t_0,a)/\varphi(B)\} = \varphi(B).\end{aligned}$$

This is a contradiction. Therefore, if $t > t_0+T(t_0,a) = t_0+1/c\, log[K(t_0,a)/\varphi(B)]$ we have $x(t) \in S_B$. Thus, the solutions of (2) are equi-ultimatelly bounded for bound B. Hence, we have the following result.

Theorem 1. Let us suppose that:

1) there exists a Lyapunov's function V(t,x) which satisfies the conditions (3)-(5), and (7),

2) the perturbing term g(t,x) satisfies the conditions (5) and (8).

Then the solutions of system (2) are equi-bounded and equi-ultimately bounded.

Remark 1. Hara in [12], studied the boundedness and asymptotic behaviour of solutions of time-variying differential equations

$$x' = A(t)x + f(t,x), \tag{12}$$

where x, f are n-vectors, A(t) is a bounded differentiable nxn matrix for $t \geq 0$, and f(t,x) is continuous in (t,x) for $t \geq 0$ and $\|x\| < \infty$. In that paper the following assumptions on f are considered

i) , $\|f(t,x)\| \leq \pi(t)(1 + \|x\|)$ with p is a non-negative continuous function on I satisfying$\int_t^{t+1} \pi(s)ds \longrightarrow 0$ as $t \longrightarrow \infty$.

In [30] the author deals with stability exponential of equation (12) using the exponential stability of linear term and a nonlinear nonstationary infinite dimensional difference equations analogous to (1).

The solutions x(t) of (1) starting from x0 at zero is given by

$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)f(s,x(s))ds,$

where F(t,s) is the fundamental solution matrix of linear system. Thus, applying a stability theorem obtained for the discrete-time system, derived the following sufficient condition for asymptotic stability of (12)

ii) [30, Th.3] Assume that

$\|\Phi(t,s)\| \leq Ke^{-\delta s},\ \forall t,s \geq 0.$

$\|f(t,x)\| \leq a(t)\, \|x\|^m,\ t \geq 0,$

if 0<m<1, $\limsup_{k \longrightarrow \infty} \int_0^1 e^{\delta(k+s)(1-m)}ds = 0,$

if m=1, $\limsup_{k \longrightarrow \infty} \int_0^1 a(k+s)ds = 0,$

if m>1, $\sum_{k=0}^{\infty} e^{-\delta(k+s)(m-1)}a(k+s)ds < \infty.$

Hara, Yoneyama and Okasaki in [13], considered the following systems of differential equations:

$$x\acute{'} = A(t)x, \tag{(L)}$$

$$y' = A(t)y + g(t,y), \tag{(LP)}$$

$$y' = A(t)y + h(t), \quad \text{((Ph))}$$

where x, y, g, h are n-vectors, A(t) is take as above, g is a continuous function on IxR^n and h is also a continuous function on I. Under conditions:

iii) $\|g(t,y)\| \leq \pi(t)\phi(\|y\|)$ for all $t \in I$ and $\|y\| \geq R$, $\int_0^\infty \pi(t)dt < \infty$ and ϕ is a positive continuous non-decreasing function on $r \geq r_0 > 0$ with $\int_0^\infty \frac{dr}{\phi(r)} = \infty$.

In that paper, they proved that if solutions of (L) are uniformly bounded (UB) and g satisfy ii), then solutions of (LP) are (UB). If solutions of (L) are (UB) and ultimately bounded (UltB) and g satisfy ii), then solutions of (LP) are (UB) and (UltB). If we consider that.

iv) $\int_0^\infty \|h(t)\|\, dt = \infty$,

then there exists a matrix A(t) such that the solutions of (L) are (UB) and (UltB) and solutions of (Ph) are not (UB). If in addition, there exists a sequence $\{t_n\}$ such that $t_n \longrightarrow \infty$ and $\left\|\int_0^{t_n} h(s)ds\right\| \longrightarrow \infty$ as $n \longrightarrow \infty$, then the matrix A(t) of the above assertion can be take bounded on I.

Hara, Yoneyama and Sugie in [14], studied the continuability of solutions of system (1) and perturbed systems (2) and

$$x' = f(t,x) + h(t), \quad \text{((2h))}$$

with f, g continuous n-functions on IxR^n and h is continuous n-vector function. Suppose that

v) $\|f(t,x)\| \leq \lambda(t)\phi(\|x\|)$ and $\|g(t,x)\| \leq \mu(t)\Psi(\|x\|)$ on IxR^n with$\int_\delta^\infty \frac{dr}{1+\Psi(r)} = \infty$ and $\Psi(r)$ is a bounded function on$r \geq \delta \geq 0$.

vi) F as above and h is such that the solutions of (2h) are of global existence (GE).

Under that conditions, they showed that the solutions of (2) and (2h) are continuable.

Notice the advantages of the condition (5) over assumptions of the perturbing term, i)-vi) above, in the referent to a priori bound of such term. Thus, a fundamental rol is played by the stronger hypothesis that (11) $\gamma = \sup_{t\geq 0} \int_t^{t+1} \gamma(s)ds < +\infty$. So, our results are obtained under milder conditions. By the other hand, the above theorem generalizes the results due to Hara [12]; Hara, Yoneyama and Okasaki [13]; and Onuchic [27], who considered the linear case $f(t,x) = A(t)x$, obtaining the boundedness property. Therefore, our result is also more general that cited above.

Remark 2. The main purpose of sections 4.6 and 4.7 of [8] is to give theorems on uniform and global asymptotic stability of the zero solution $x \equiv 0$ of the system (12), under assumption of type

$$\|f(t,x)\| \leq L(t,\|x\|),\ t \in [a,b), x \in R^n$$

with

$0 \leq L(t,u) - L(t,v) \leq M(t,v)(u-v), t \in [\alpha,\beta),\ u{\geq}v{\geq}0$, M is nonnegative and continuous on $[\alpha,\beta)xR^n$.

These results are obtained, in general, with additional integral boundedness conditions on M. It is clear that using ideas presented here we can derive results on uniform and global asymptotic stability of solution $x \equiv 0$ of (12) under milder assumptions (see Theorems 2 and 3 below).

Remark 3. Without use of Theorem 10.1 of [34], the Theorem 1 only cover the asymptotic stability (see, for example, pioneer results of this nature in Corduneanu [5] and Germanidze [11]). A result of this type, is consistent with the results obtained in [13] for asymptotically autonomous differential equations on a plane (and with a stronger version of Markus´ theorem [19], given in [1]), with the results of section 3 of [17] and with [24, Theorem 1.6].

Remark 4. Furumochi [10], employing Lyapunov's second method, discussed the uniform asymptotically stability of the zero solution of (2), including the case when f satisfies weaker assumptions than those reported in the references. Later, using a result of Chow and Yorke [4], on integral attraction of zero solution of (1), obtained the integral attractivity of zero solution of perturbing equation (2), under suitable assumptions. So, taking in account the above remark, our Theorem 1 complete the study of asymptotic stability of zero solution of perturbing equation (2), under appropiate assumptions. That remark still valid if we take in account [9].

Remark 5. In [23] Redheffer studied the Lotka-Volterra system

$$x_i' = x_i\left(e_i + \sum_{j=1}^{m} p_{ij}x_j\right),\ x_i(0) > 0,\ 0 \leq t < +\infty$$

where $p = (p_{ij})$ is a real m by m matrix and where the equations are to hold for i=1,2,...,m. It is assumed that the constants ei are so choosen that a stationary point $q = (q_i)$ in the first quadrant exists. By that study is considered an equation of the equation of the form $x' = f(x) + h(t)$, a simple case of problem (1)-(2). If $v : R \longrightarrow R$ is a Liapunov function for f, the function $R \longrightarrow R$ defined by $V(t) = v(x(t))$ satisfies, not the usual condition

$$V'(t) = \tfrac{d}{dt}v(x(t)) = (gradv(x))f(x) \leq 0$$

along trajectories, but the weaker condition $V'(t) \leq g(t)$, where the error term

$g(t) = (gradv(x))h(t)$

itself depends on the unknown solution x (see (7) above). The author assume that the integral

$$\int_0^\infty g(s)ds = \underset{t\longrightarrow\infty}{Lim} \int_0^t g(s)ds = L$$

exists as a finite value. Let Λ^+ be the w-limit set for a given trajectory, x(t), which remains fixed throughout the study. Redheffer obtained the following result.

Lemma 2. Under the above conditions, the trajectories are bounded and v(x) is constant on Λ^+.

So, it is clear that the above result is equivalent to our Theorem 1.

The following theorems are variant of the above and are essentially generalizations of results due to de Molfetta [20], Hara [12] and Onuchic [27] in the linear case.

Theorem 2. Consider the systems (1) and (2) where f and g are continuous on D. Suppose in addition to conditions (5)-(7) and (9) that the Lyapunov function V satisfied the following hypothesis:

$$a(x) \leq V(t,x) \text{ and } V'_{(1)}(t,x) \leq -\lambda(t)V(t,x) - \mu(t)W(t,x), \tag{13}$$

with a(r) a continuous and positive defined function satisfying $a(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ and W is some defined positive function on IxS_r. Then the solution $x \equiv 0$ of (2) is equi-asymptotically stable in the whole.

Proof. Under (13), we proved in [24] that solution $x \equiv 0$ of (1) is equi-asymptotically stable in the whole.

As in the proof of the above theorem, we have that (11) still valid. So, we have that $V(t; x(t; t_0, x_0)) \leq V(t_0, x_0)exp(-L(t))$, with $L(t)$ as above. Let $M(t_0, \varepsilon) = \max_{\|x\|<\varepsilon} V(t_0, x_0)$ and $T(t_0, t, \varepsilon)$ such that $T(t_0, t, \varepsilon) = L^{-1}\left(\ln \frac{M(t_0,\varepsilon)}{a(\tau)}\right)$. Consequently, for $t > t_0 + T(t_0, t, \varepsilon)$ we obtain that $V(t; x(t; t_0, x_0)) < a(\tau)$.

¿From assumptions (13) it follows that for $\|x(t; t_0, x_0)\| < \tau$ for $t > t_0 + T(t_0, t, \varepsilon)$, which means the equi-asymptotically stability in the whole of solution $x \equiv 0$ of system (2).■

By using Theorem 2 of [24] on the asymptotic stability in the whole of zero solution of system (1), and a similar idea to the above proof, we can prove the following result.

Theorem 3. Consider the systems (1) and (2) where f and g are continuous on D. Suppose in addition to conditions (5)-(7) and (9) that the Lyapunov function satisfy the following hypotheses:

$$V'_{(1)}(t,x) \leq -\lambda(t)\phi(V(t,x)) - \mu(t)W(t,x),$$

where $f \in CC(R^+)$ and W is some defined positive function on IxS_r. Then the solution $x \equiv 0$ of system (2) is asymptotically stable in the whole.

Remark 6. Our results generalize, in particular, those of [20] and [27], who under the same conditions on the functions f and g, obtains only the asymptotic stability property.

Remark 7. If in systems (1) and (2) we have that $f(t,x) = f(x)$, our results are consistent with obtained by Hutson [15].

Remark 8. During the last forty years, investigators have shown that if solutions of general differential equations (1) are uniformly bounded (UB) and uniformly ultimately bounded (UUB) and if $x(t+T)$ is a solution whenever $x(t)$ is a solution. Much less progress has been made in developing general techniques which imply that solutions are UB and UUB. It has seemed that Lyapunov's second method is an appropiate vehicle for this endeavor, but that path has been marked by persistent difficulties. Frecuently (2) has the form

$$x' = f(t,x) + g(t),$$

where $g(t+T) = g(t)$ and (1) has the zero solution. In [3] the author conjectured that if (1) enjoyed strong global asymptotic stability, then solution of above equation should be UB and UUB. It is clear that our results is a partial answer to that question.

3. Some applications and comparisons.

1) From the above theorems and results of [23], [25] and [26] (also cf. [22]), we can obtain some asymptotic properties of a perturbed bidimensional system, which contains the classical Liénard equation:

$$x'' + f(x)x' + a(t)g(x) = 0. \tag{14}$$

Thus, let

$$x' = \alpha(y) - \beta(y)f(x),\ y' = -a(t)g(x), \tag{15}$$

where the continuous functions involved satisfying:

i. $\alpha \in CI(R)$,
ii. $\beta \in CP(R)$ and $\beta \geq \underline{b} > 0$ for all y,
iii. $f, g \in CC(R)$ and $G(x) = \int_0^x g(r)dr \leq f(x)g(x)$, for all x,
iv. $a \in C^1P(I)$ and bounded, i.e., and $0 < \underline{a} \leq a(t) \leq \overline{a} < \infty$ and $a'(t) \geq c > 0$, $c - \underline{ab} < 0$.

Under this assumptions, we proved in [24] the asymptotic stability in the whole of zero solution of system (15). Taking the perturbed system:

$$x' = \alpha(y) - \beta(y)f(x) + h_1(t,x,y), y' = -a(t)g(x) + h_2(t,x,y), \tag{16}$$

With the Lyapunov function

$$V(t,x,y) = A(y) + a(t)G(x), A(y) = \int_0^y \alpha(s)ds.$$

Making ,$\|z\| = A(y) + kG(x)$, $z = (x, y)$ and k is some positive constant we have:

Theorem 4. Under i-iv assumptions, if h satisfies (7), with (10) and (11), then the zero solution of system (16) is stable asymptotically in the whole.

Proof. Is enough to prove that all conditions of Theorem 1 hold and consider the Remark 3.■

2) Analogous result on the continuability of solutions of Liénard equation (16) under perturbations can be obtained, taking in account the paper [22] and idea presented here, without make use of system (17).

3) The main result of [32] is a continuation theorem in the spirit of degree theoretic continuation theorems, for the existence of bounded solutions of ordinary differential equations of type (1) with $f(t,x) = f(x)$, i.e.

$$x' = f(x) + g(t,x), \tag{17}$$

where f is a homogeneous function of degree p at least one. The author assume $x = 0$ is the only bounded solution (with $h(\beta, \{0\}) \neq 0$) of

$x' = f(x)$,

and

$$\underset{|x|\to\infty}{Lim} \frac{g(t,x)}{|x|^p} = 0 \tag{18}$$

and obtain that (17) has a full bounded solution.

In [31] a related result is proved for differential equations of the form (1). It is assumed that $f(t,x)$ is homogeneous in x of degree $p > 1$, $g(t,x)$ satisfies the growth (19) of earlier result, and both functions are uniformly almost periodic in t. Thus the conditions here on g are less restrictive, so that in the case of autonomous f, equation (18), the result here is much more general. Moreover, one could prove a more encompassing version of the above result, taking into account the Theorem 1.

4) Marlin and Struble [18, Corollary 1] proved that the system (LP) and (L), with $A(t) = (a_{ij}(t))_{i,j=1,\dots,n}$ are asymptotically equivalent under the following main assumptions:

a) there exist continuous scalar function $G_k(t)$, $k = 1, ..., n$ such that $|g_k(t,x)| \leq G_k(t)$, $(t,y) \in IxR^n$.

b) $\underset{t\to\infty}{Lim} \int_t^\infty G_k(v) \left[\exp \int_c^t a_{kk}(s)ds\right] dv = 0$, $k = 1, ..., n$.

c) $\underset{t\to\infty}{Lim} \int_t^\infty |a_{ij}(v)| \left[\exp \int_c^t a_{ii}(s)ds\right] dv = 0, i, j = 1, ..., n, i \neq j$.

We note that the conditions of our results are independent of the above conditions. If we take j^{o}1, it is clear that earlier result, can be obtained under milder assumptions.

In [7], Díblik considered systems:

$$y' = A(x)y + a(x,y,z), z' = B(x)z + b(x,y,z), \tag{19}$$

and

$$\overline{y}' = A(x)\overline{y}, \overline{z}' = B(x)\overline{z} \tag{20}$$

where the matrices $A(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ and $B(x) = (b_{ij}(x))_{i,j=1,\dots,n}$ are real and continuous on I, $a(x,y,z)$ and $b(x,y,z)$ are real and continuous k-vector and continuous s-vector on region IxR^nxR^k. Moreover, it is assumed the existence and continuous dependence of the solutions of (20) in the interval I. Under this assumptions showed that the systems (20) and (21) are k-asymptotically equivalent on I, i.e., for every solution $(y(x), z(x))$ of (20) there is at least a k-parametric family of solutions of (21), and conversely, such that

$$\underset{x\to\infty}{Lim}\,(y_i(x)-\overline{y}_i(x))=0, i=1,...,k$$

and

$$\underset{x\to\infty}{Lim}\,(z_i(x)-\overline{z}_i(x))=0, i=i,...,s.$$

It is clear that the results of [7] can be formulated in terms of Theorem 1.

If we take in (1) and (2) $f(t,x)=Ax+p(t)$, i.e.,

$$x'=Ax+p(t), \tag{21}$$

$$x'=Ax+p(t)+g(t,x), \tag{22}$$

where A is a nxn constant matrix, $p(t)$ is continuous on I and $g(t,x)$ is continuous on IxR^n. The following assumptions are considered in [34]:

i) all solutions of (22) are bounded,

ii) $\|f(t,x)\| \le \lambda(t)\varphi(\|x\|)$ where $\lambda(t)\geq 0$ is continuous, $\int_0^\infty \lambda(t)dt < \infty$ and $\varphi(r)>0$ for $r\geq 0$, $\varphi(r)$ is continuous, increasing and condition (10) holds.

Then, the systems (22) and (23) are asymptotically equivalent, that is, for any given solution of (22) there exists a solution of (23) which approaches the given solution of (22) as $t \longrightarrow \infty$, and conversely. It is clear that our conclusions are equivalent to Yoshizawa's conclusion (see [34, Lemma 24.1] for other result quite different) and our hypotheses are stronger than his. But on the way we can establish several results of independent interest and may prove useful in other applications.

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