SOME INEQUALITIES INVOLVING THE GEOMETRIC MEAN OF NATURAL NUMBERS AND THE RATIO OF GAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

Abstract. In this article, using Stirling's formula, the series-expansion of digamma functions and other techniques, two inequalities involving the geometric mean of natural numbers and the ratio of gamma functions are obtained.

1. Introduction

In [1], Dr. H. Alzer proved that the inequalities

$$\frac{n+2\sqrt{2}-1}{n+1} \le \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+2}{n+1} \tag{1}$$

hold for all integers $n \ge 1$. The lower and upper bounds in (1) are the best possible.

He also verified in [2] that the inequality

$$\frac{[\Gamma(x+2)]^{1/(x+1)}}{[\Gamma(x+1)]^{1/x}} < \frac{x+2}{x+1} \tag{2}$$

holds for $x \ge 2$.

Since $\Gamma(n+1) = n!$, the right hand side in (1) can be deduced from inequality (2) only if we let $x = n \ge 2$. Moreover, the right hand side in (1) refines the inequality

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+1}{n}, \tag{3}$$

which was obtained in [12] by H. Minc and L. Sathre.

Recently, in [18] and [22], the first author obtained the following

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < \sqrt{\frac{n+k}{n+m+k}} \tag{4}$$

for positive integers n and m and nonnegative integer k.

The inequality (3) was refined by Dr. H. Alzer in [3]: Let $n \in \mathbb{N}$, then, for any $r > 0$, we have

$$\frac{n}{n+1} \le \left(\frac{1}{n}\sum_{i=1}^{n} i^r \bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{5}$$

The lower and upper bounds are the best possible.

Many new and simple proofs of the inequalities in (5) and some generalizations were given in [5, 6, 10, 11, 14, 17, 21, 23, 24, 27].

Date: September, 2000.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D07.

Key words and phrases. Inequality, geometric means, natural number, gamma function, Stirling's formula, digamma function.

The authors were supported in part by NSF of Henan Province (#004051800), SF for Pure Research of the Education Department of Henan Province (#1999110004), Doctor Fund of Jiaozuo Institute of Technology, and NNSF (#10001016) of China.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

The left hand side of inequality (5) was generalized in [16]: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k}<\left(\frac{1}{n}\sum_{i=k+1}^{n+k}i^r\Bigg/\frac{1}{n+m}\sum_{i=k+1}^{n+m+k}i^r\right)^{1/r},\tag{6}$$

where r is any given positive real number. The lower bound is the best possible.

The integral analogue of (6) was presented in [8] and [15]: Let $b>a>0$ and $\delta>0$ be real numbers, then, for any given positive $r\in\mathbb{R}$, we have

$$\begin{aligned}\frac{b}{b+\delta}&<\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}\\&=\left(\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x\Bigg/\frac{1}{b+\delta-a}\int_a^{b+\delta}x^r\,\mathrm{d}x\right)^{1/r}\\&<\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}.\end{aligned}\tag{7}$$

The lower and upper bounds in (7) are the best possible.

The inequality (7) was generalized to an inequality for linear positive functionals in [7].

Recently, results related to those above were obtained in [19]. These results were generalisations for monotonic sequences involving convex functions as follows:

- For $a>1$, let $n\in\mathbb{N}$ and $r>0$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n}a^{ir}\Bigg/\frac{1}{n+1}\sum_{i=1}^{n+1}a^{ir}\right)^{1/r}>\frac{1}{a}.\tag{8}$$

- For $n,m\in\mathbb{N}$, $k\in\mathbb{N}\cup\{0\}$ and $r>0$, we have

$$\frac{1}{a^m}<\left\{\frac{1}{a^n}\sum_{i=k+1}^{n+k}a^{ir}\Bigg/\frac{1}{a^{n+m}}\sum_{i=k+1}^{n+m+k}a^{ir}\right\}^{1/r},\tag{9}$$

  that is,

$$\frac{1}{a^{m(r+1)}}\le\sum_{i=k+1}^{n+k}a^{ir}\Bigg/\sum_{i=k+1}^{n+m+k}a^{ir},\tag{10}$$

  where $a>1$ is a positive real number.
- If $\{a_i\}_{i\in\mathbb{N}}$ is an increasing, positive sequence such that $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases, then we have

$$\frac{a_n}{a_{n+1}}\le\sqrt[n]{\prod_{i=1}^{n}(a_i+a_n)}\Bigg/\sqrt[n+1]{\prod_{i=1}^{n+1}(a_i+a_{n+1})}\le\sqrt[n]{\prod_{i=1}^{n}a_i}\Bigg/\sqrt[n+1]{\prod_{i=1}^{n+1}a_i}.\tag{11}$$

- If φ is an increasing, convex, positive function defined on $(0,\infty)$ such that $\left\{\varphi(i)\left[\frac{\varphi(i)}{\varphi(i+1)}-1\right]\right\}_{i\in\mathbb{N}}$ decreases, then

$$\frac{[\varphi(n)]^{n/\varphi(n)}}{[\varphi(n+1)]^{(n+1)/\varphi(n+1)}}\le\sqrt[\varphi(n)]{\prod_{i=1}^{n}[\varphi(i)+\varphi(n)]}\Bigg/\sqrt[\varphi(n+1)]{\prod_{i=1}^{n+1}[\varphi(i)+\varphi(n+1)]}.\tag{12}$$

These inequalities generalize those obtained in [10], [17], and [22].

In this article, we will prove the following inequalities

Theorem 1. *For $m, n \in \mathbb{N}$ and nonnegative integer k, we have*

$$\frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} > \frac{n+k+1}{n+m+k+1}. \tag{13}$$

Theorem 2. *The function*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{14}$$

is decreasing in $x \ge 1$ for fixed $y \ge 0$.

If the above hold, then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \le \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{15}$$

Remark 1. If we take $x, y \in \mathbb{N}$, then the right hand side of (4) and inequality (13) follow from (15).

2. Proofs of Theorems

Proof of Theorem 1. Inequality (13) can be rearranged so that we have

$$\frac{n+k+1}{\sqrt[n]{(n+k)!/k!}} < \frac{n+m+k+1}{\sqrt[n+m]{(n+m+k)!/k!}},$$

which is equivalent to

$$\frac{n+k+1}{\sqrt[n]{(n+k)!/k!}} < \frac{n+k+2}{\sqrt[n+1]{(n+k+1)!/k!}}. \tag{16}$$

When $k = 0$, inequality (16) follows from the right inequality in (1).

When $k \ge 1$, the inequality (16) can be rewritten as

$$\left[\frac{(n+k)!}{k!}\right]^{1/n} > \frac{(n+k+1)^{n+2}}{(n+k+2)^{n+1}}. \tag{17}$$

In [10] and [13, p. 184], the following inequalities were given for $n \in \mathbb{N}$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{1}{12n}. \tag{18}$$

Inequality (18) is related to the Stirling's formula. In [25], Professor J. Sándor and L. Debnath proved a new form of the Stirling's formula: For all positive integers $n \ge 2$, we have the double inequality

$$\sqrt{2\pi}\, \mathrm{e}^{-n}\, n^{n+1/2} < n! < \left(\frac{n}{n-1}\right)^{1/2}\sqrt{2\pi}\, \mathrm{e}^{-n}\, n^{n+1/2}. \tag{19}$$

By substituting the inequalities in (18) into the left term of inequality (17), we see that it is sufficient to prove the following

$$\left[\sqrt{2\pi(n+k)}\left(\frac{n+k}{e}\right)^{n+k}\right]^{1/n} > \frac{(n+k+1)^{n+2}}{(n+k+2)^{n+1}}\left[\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \exp\frac{1}{12k}\right]^{1/n}. \tag{20}$$

Taking the logarithm on both sides of inequality (20), simplifying directly and using standard arguments, we obtain

$$\frac{2k+1}{2n}\ln\left(1+\frac{n}{k}\right)+(n+1)\ln\left(1+\frac{1}{n+k+1}\right)-\ln\left(1+\frac{1}{n+k}\right)-\frac{1}{12kn}-1 > 0. \tag{21}$$

In [9, pp. 367–368], [13, pp. 273–274] and [20], we have for $t > 0$

$$\ln\left(1+\frac{1}{t}\right) > \frac{2}{2t+1}, \tag{22}$$

$$\ln(1+t) < \frac{t(2+t)}{2(1+t)}. \tag{23}$$

Thus, to get inequality (21), it suffices to show that

$$\frac{2(n+1)}{2(n+k+1)+1} + \frac{2k+1}{2k+n} - \frac{2(n+k)+1}{2(n+k)(n+k+1)} - \frac{1}{12kn} - 1 > 0,$$

which can be deduced from the following

$$\begin{aligned}
&12kn^4(k-1) + 2n^3(n+5k)(k^2-1) + 5n^3(k^3-1) + 6k^2n^2(kn-1)\\
&+ 3n^2(k^3n-1) + 2k(n^2+3k)(k^2n-1) + k^2n(kn^2-1)\\
&+ 9kn(k^2n^2-1) + 10k^3(n^3-1) + 2k^2n(k+12)(n^2-1)\\
&+ 4k^4(6n^2-1) + 6k^3n(3k+10n) + 10k^2n^4 > 0.
\end{aligned} \tag{24}$$

The proof is complete. □

Proof of Theorem 2. For a fixed real number $y \ge 0$, define

$$w(x) = \frac{\ln\Gamma(x+y+1) - \ln\Gamma(y+1)}{x} - \ln(x+y+1), \quad x \in [1,\infty). \tag{25}$$

A simple calculation reveals that

$$w'(x) = \frac{\ln\Gamma(y+1) - \ln\Gamma(x+y+1)}{x^2} - \frac{1}{x+y+1} + \frac{\psi(x+y+1)}{x}, \tag{26}$$

where $\psi = \Gamma'/\Gamma$ denotes the logarithmic derivatives of the gamma function. It is also called a digamma function in [4, p. 71].

It is well-known that

$$\Gamma(z+1) = z\Gamma(z), \quad \operatorname{Re}(z) > 0; \tag{27}$$

$$\psi(x) < \ln x - \frac{1}{2x}, \quad x > 1; \tag{28}$$

$$\psi'(z) = \sum_{i=0}^{\infty} \frac{1}{(i+z)^2};. \tag{29}$$

The inequality (28) can be found in [9, 12, 13] respectively. For more on formula (29), please refer to formula (8.12) in Theorem 8.3, page 93 in [26].

Using the formulae (27) and (29) and inequalities (23) and (28) and direct computation, we have

$$\begin{aligned}
\frac{[x^2w'(x)]'}{x} &= \psi'(x+y+1) - \frac{x+2y+2}{(x+y+1)^2}\\
&= \sum_{i=1}^{\infty} \frac{1}{(x+y+i)^2} - \frac{x+2y+2}{(x+y+1)^2}\\
&< \frac{1}{(x+y+1)^2} + \int_1^{\infty} \frac{\mathrm{d}t}{(x+y+t)^2} - \frac{x+2y+2}{(x+y+1)^2}\\
&= -\frac{y}{(x+y+1)^2}\\
&< 0,
\end{aligned} \tag{30}$$

and

$$\begin{aligned} w'(1) &= \ln\Gamma(1+y) - \ln\Gamma(2+y) + \psi(2+y) - \frac{1}{2+y} \\ &= \psi(2+y) - \ln(1+y) - \frac{1}{2+y} \\ &< \ln(2+y) - \ln(1+y) - \frac{1}{2(2+y)} - \frac{1}{2+y} \\ &= \ln\left(1+\frac{1}{1+y}\right) - \frac{3}{2(2+y)} \\ &< \frac{2y+3}{2(1+y)(2+y)} - \frac{3}{2(2+y)} \\ &= -\frac{y}{2(1+y)(2+y)} \\ &< 0. \end{aligned} \tag{31}$$

Thus, the function $x^2 w'(x)$ is decreasing, $x^2 w'(x) < w'(1) < 0$, and the function $w(x)$ is decreasing with $x > 1$. That is, the function $\left[\Gamma(x+y+1)/\Gamma(y+1)\right]^{1/x}/(x+y+1)$ is decreasing with $x > 1$ for fixed $y \geq 0$. This completes the proof. □

References

[1] H. Alzer, *On some inequalities invoving* $(n!)^{1/n}$, Rocky Mountain J. Math. **24** (1994), no. 3, 867–873.

[2] H. Alzer, *On some inequalities invoving* $(n!)^{1/n}$*, II*, Period. Math. Hungar. **28** (1994), no. 3, 229–233.

[3] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[4] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathyematics **97**, Addison Wesley Longman Limited, 1998.

[5] S. S. Dragomir and J. van der Hoek, *Some new analytic inequalities and their applications in guessing theory*, J. Math. Anal. Appl. **225** (1998), 542–556.

[6] N. Elezović and J. Pečarić, *On Alzer's inequality*, J. Math. Anal. Appl. **223** (1998), 366–369.

[7] B. Gavrea and I. Gavrea, *An inequality for linear positive functionals*, J. Ineq. Pure Appl. Math. **1** (2000), no. 1, Article 5. `http://jipam.vu.edu.au/v1n1/004_99.html`.

[8] B.-N. Guo and F. Qi, *An Algebric Inequality, II*, RGMIA Res. Rep. Coll. **4** (2001), no. 1. `http://rgmia.vu.edu.au/v4n1.html`.

[9] J.-C. Kuang, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[10] J.-C. Kuang, *Some extensions and refinements of Minc-Sathre inequality*, Math. Gaz. **83** (1999), 123–127.

[11] J. S. Martins, *Arithmetic and geometric means, an applications to Lorentz sequence spaces*, Math Nachr. **139** (1988), 281–288.

[12] H. Minc and L. Sathre, *Some inequalities involving* $(n!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41–46.

[13] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York/Heidelberg/Berlin, 1970.

[14] N. Ozeki, *On some inequalities*, J. College Arts Sci. Chiba Univ. **4** (1965), no. 3, 211–214. (Japanese)

[15] F. Qi, *An algebraic inequality*, J. Ineq. Pure Appl. Math. **2** (2001), in the press. `http://jipam.vu.edu.au/006_00.html`. RGMIA Res. Rep. Coll. **2** (1999), no. 1, Article 8, 81–83. `http://rgmia.vu.edu.au/v2n1.html`.

[16] F. Qi, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. **240** (1999), 294–297.

[17] F. Qi, *Generalizations of Alzer's and Kuang's inequality*, Tamkang J. Math. **31** (2000), no.3, in the press. RGMIA Research Report Collection **2** (1999), no. 6, Article 12. `http://rgmia.vu.edu.au/v2n6.html`.

[18] F. Qi, *Inequalities and monotonicity of sequences involving* $\sqrt[n]{(n+k)!/k!}$, RGMIA Res. Rep. Coll. **2** (1999), no. 5, Article 8. `http://rgmia.vu.edu.au/v2n5.html`.

[19] F. Qi, *Monotonicity of sequences involving convex function and sequence*, RGMIA Res. Rep. Coll. **3** (2000), no. 2, Article 14. `http://rgmia.vu.edu.au/v3n2.html`.

[20] F. Qi and C.-P. Chen, *Monotonicity of two sequences*, Mathematics and Informatics Quarterly **9** (1999), no. 4, 136–139.

[21] F. Qi and L. Debnath, *On a new generalization of Alzer's inequality*, Internat. J. Math. & Math. Sci. **23** (2000), no. 12, 815–818.

[22] F. Qi and Q.-M. Luo, *Generalization of H. Minc and J. Sathre's inequality*, Tamkang J. Math. **31** (2000), no. 2, 145–148. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Article 14. `http://rgmia.vu.edu.au/v2n6.html`.

[23] J. Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034–1035.

[24] J. Sándor, *Comments on an inequality for the sum of powers of positive numbers*, RGMIA Res. Rep. Coll. **2** (1999), no. 2, 259–261. `http://rgmia.vu.edu.au/v2n2.html`.

[25] J. Sándor and L. Debnath, *On certain inequalities involving the constant e and their application*, J. Math. Anal. Appl. **249** (2000), no. 2, 569–592.

[26] Lin Tan, *Reading Notes on the Gamma Function*, Zhejiang University Press, Hangzhou City, Zhejiang, China, 1997. (Chinese)

[27] J. S. Ume, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), no. 3, 521–522.

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China

E-mail address: `qifeng@jzit.edu.cn`

URL: `http://rgmia.vu.edu.au/qi.html`

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China