Hadamard Type Inequality for Quasiconvex Functions In Higher Dimensions *

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Abstract

In this article we study a Hadamard type inequality for nonnegative evenly quasiconvex functions. The approach of our study is based on the notion of abstract convexity. We also provide an explicit calculation to evaluate the asymptotically sharp constant asscociated with the inequality over a unit square in the two dimensional plane.

Key words quasiconvex functions, Hadamard Inequality, Supremal generators.

1 Introduction

The well-known Hadamard inequality (see, for example, [1, 2] and references therein) asserts that for convex bounded function f defined on the segment $[a, b]$ of the real line $\mathbb{R}$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}(f(a)+f(b)). \tag{1}$$

In this paper we shall study generalizations of the left side of this inequality for some classes of quasiconvex functions defined on a convex subset of n- dimensional space $\mathbb{R}^n$. Generalization of the left side of (1) for convex functions defined on a convex subset of $\mathbb{R}^n$ are well-known. For example if $X \subset R^n$ is a convex bounded symmetrical set (the latter means that $x \in X \implies -x \in X$), then (see, for example, [5])

$$f(0) \leq \frac{1}{\mu(X)}\int_X f(x)dx. \tag{2}$$

for each lower semicontinuous convex function $f : X \to \mathbb{R}$, where $\mu(X)$ is the volume of the set X.

Generalizations of (1) for quasiconvex functions defined on the real line also well-known. It was established in [2], see also [4, 5] that for a quasiconvex function f defined on $[a, b]$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \tag{3}$$

All discussed versions of the Hadamard type inequality give an estimate of the integral through the centre of the central symmetric set. A version of the Hadamard type inequality with respect to an arbitrary interior point of the segment on the real line was established in [4]. In particular, it was shown at [4] that for an arbitrary point $u \in (0, 1)$ and arbitrary nonnegative quasiconvex function define on $[0, 1]$ the following inequality holds:

$$f(u) \leq \frac{1}{\min(u, 1-u)}\int_0^1 f(x)dx \tag{4}$$

In the present paper we extend an approach developed in [4] for some classes of nonnegative quasiconvex functions defined on the closed convex subset X of a n-dimensional space $\mathbb{R}^n$. We show that the asymptotically sharp estimate γ in the inequality $f(u) \le \gamma \int_X f(x)d\mu$, where f is an evenly quasiconvex nonnegative function defined on X, can be obtained through the calculation of the measure of certain slices of the set X. (Here μ is an arbitrary finite Borel measure on X.) Then we consider the inequality of the form

$$f(u) \le \gamma_h \int_X h(f(x))d\mu, \qquad f \in Q_0(X)$$

where $Q_0(X)$ is the set of evenly quasiconvex nonnegative functions f defined on X such that $f(0) = 0$ (it is assumed that $0 \in X$). Here $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that $\lambda_h := \sup_{c>0}(c/h(c)) < +\infty$. We show that the asymptotically sharp estimate g_h in this inequality is the product $\lambda_h \cdot \gamma_*$ we γ_* is the asymptotically sharp constant in the inequality $f(u) \le \gamma_* \int_X f(x)d\mu$, $f \in Q_0(X)$.

Finally, we give an explicit expression of the constant γ_* for the case, where X is the unit square on the plain $\mathbb{R}^2$.

2 Preliminaries

Let Y be a set of real-valued functions defined on a set X. We assume that Y is equipped with the point-wise order relation: if $f, g \in Y$ then $f \ge g$ if and only if $f(x) \ge g(x)$ for all $x \in X$. A set $L \subset Y$ is called a supremal generator of Y if $f(x) = \sup\{l(x) : l \in L, l \le f\}$ for all $x \in X$. Our study of Hadamard -type inequalities is based on the following *Principle of preservation of inequalities* (see [3] and also [5])

Proposition 1 *Let L be a supremal generator of Y and let ψ be an increasing functional defined on Y, that is $(f, g \in Y, f \ge g) \implies \psi(f) \ge \psi(g)$. Let further, $u \in X$. Then*

$$(l(u) \le \psi(l) \quad \textit{for all} \quad l \in L) \implies f(u) \le \psi(f) \quad \textit{for all} \quad f \in Y).$$

For the sake of completeness we present the simple proof of this Proposition. Let $l \in U := \{l' \in L : l' \le f\}$. Since $l \le f$ and ψ is an increasing functional, we have $l(u) \le \psi(l) \le \psi(f)$. Hence, $f(u) = \sup_{l \in U} l(u) \le \psi(f)$. Thus the result follows.

Let L, Y, ψ and u be as in Proposition 1. Assume that Y consists of nonnegative functions and $\psi(y) \ge 0$ for all $y \in Y$. We accept the following convention:

$$\frac{0}{0} = 0, \qquad \frac{c}{+\infty} = 0 \quad \text{for all} \quad c \ge 0, \frac{c}{0} = +\infty \quad \text{for all} \quad c > 0.$$

Let

$$\gamma = \sup_{l \in L} \frac{l(u)}{\psi(l)}. \tag{5}$$

Assume that $\gamma < +\infty$. Then, due to Principle of Preservation of Inequalities, we have

$$f(u) \le \gamma\psi(f) \qquad \text{for all} \quad f \in Y. \tag{6}$$

Note that γ is the *asymptotically sharp* constant. This means that for each $\varepsilon > 0$ there exists a function $f \in Y$ such that the inequality $f(u) \le (\gamma - \varepsilon)\psi(f)$ does not hold. Indeed, the asymptotic sharpness follows directly from the definition of the supremum. Thus in order to establish the inequality of the form (6) with the asymptotically sharp constant we need to calculate the number γ given by (5). It can be proved in some instances that there exists $f \in Y$ such that $f(u) = \gamma\psi(f)$. If this inequality holds then the constant γ is called *sharp*.

3 Hadamard type inequality for quasiconvex functions

Recall that a function f defined on a convex set X in a n-dimensional space $\mathbb{R}^n$ is called quasiconvex if $f(\alpha x + (1-\alpha y) \le \max(f(x), f(y))$ for all $x, y \in X$ and $\alpha \in (0,1)$. An equivalent definition: a function f is called quasiconvex if its level sets $S_c(f) = \{x : f(x) \le c\}$ are convex for all $c \in R$. It is well-known that a function f of the form $f(x) = g(x)/h(x)$ is quasiconvex if g is nonnegative and convex and h is positive and concave. A set X is called evenly quasiconvex if for each $x' \notin X$ there exists $v \in \mathbb{R}^n$ such that $[v, x'] > [v, x]$ for each $x \in X$. (Here and in the sequel $[v, x]$ stands for the inner product of vectors v and x.) Clearly each open convex set and closed convex set are evenly convex. A function f is called evenly quasiconvex if its level sets $S_c(f)$ are evenly convex for all c. Each quasiconvex function, which is either lower semicontinuous or upper semicontinuous, is evenly quasiconvex.

Denote by Q^+ the set of all nonnegative evenly quasiconvex functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty} \equiv \mathbb{R} \cup \{+\infty\}$. Let $Q_0^+ = \{f \in Q^+ : f(0) = 0\}$. For each vector $v \in \mathbb{R}^n$ and each number $c \ge 0$ consider the function $l^{v,c}$ defined by

$$l^{v,c}(x) = \begin{cases} c & \text{if } [v,x] \ge 1 \\ 0 & \text{if } [v,x] < 1. \end{cases} \tag{7}$$

Let $L_0 = \{l^{v,c} : v \in \mathbb{R}^n, c \in \mathbb{R}_+\}$. It is easy to check that $L_0 \subset Q_0^+$. The following result hold (see, for example, [5]).

Proposition 2 *L_0 is a supremal generator of Q_0^+.*

Let $X \subset \mathbb{R}^n$ be a closed convex set. Denote by $Q^+(X)$ the set of all nonnegative evenly quasiconvex functions $f : X \to \mathbb{R}_{+\infty}$ such that the minimum $\min\{f(x) : x \in X\}$ is attained. If X is compact then each lower semicontinuous function f defined on X belongs to $Q^+(x)$. We now define a set $L(X)$ of two-step functions defined on X. A function l belongs to $L(X)$ if and only if there exists a vector $v \in \mathbb{R}^n$, a point $x_0 \in X$ and nonnegative numbers c and d such that $c \ge d$ and for all $x \in X$ we have:

$$l(x) = \begin{cases} c & \text{if } [v, x - x_0] \ge 1 \\ d & \text{if } [v, x - x_0] < 1. \end{cases} \tag{8}$$

Clearly $L(X) \subset Q^+(X)$.

Proposition 3 *The set $L(X)$ is a supremal generator of the set $Q^+(X)$.*

Proof: Let $f \in Q^+(X)$. Then there exists a point $x_0 \in X$ such that $f(x) \ge f(x_0)$ for all $x \in X$. Consider the set $Z = X - x_0$ and the function g defined on Z by

$$g(z) = \begin{cases} f(z + x_0) - f(x_0) & \text{if } z \in Z \\ +\infty & \text{if } z \notin Z. \end{cases}$$

Clearly $g \in Q^+$. Due to Proposition 2, there exists a set $U \subset L_0$ such that $g(z) = \sup_{l \in U} l(z)$ for all $z \in \mathbb{R}^n$. For each $l \in U$ consider the function $m(l)$ defined on X by $m(l)(x) = l(x - x_0) + f(x_0)$. It is easy to check that $m(l) \in L(X)$ and $f(x) = \sup_{l \in U} m(l)(x)$ for all $x \in X$. $\triangle$

Let

$$\psi(f) = \int_X f(x)d\mu, \qquad (f \in Q^+(X)) \tag{9}$$

where μ is a finite Borel measure defined on X. Let $u \in X$. We wish to establish the asymptotically sharp inequality of the form

$$f(u) \leq \gamma \int_X f(x)d\mu,$$

which is valid for all $f \in Q^+(X)$. For this purpose we need to calculate the constant γ defined by (5). In our case

$$\gamma = \sup_{l \in L(X)} \frac{l(u)}{\int_X l(x)d\mu}. \tag{10}$$

Let $u \in X$. In order to calculate γ, we need the following sets:

$$A_u^+ = \{(v, x_0) \in \mathbb{R}^n \times X : [v, u - x_0] \geq 1\} \tag{11}$$

and

$$A_u^- = \{(v, x_0) \in \mathbb{R}^n \times X : [v, u - x_0] < 1\}. \tag{12}$$

We also define for each $v \in \mathbb{R}^n$ and $x_0 \in X$ sets

$$X_{v,x_0}^+ = \{x \in X : [v, x - x_0] \geq 1\}, \qquad X_{v,x_0}^- = \{x \in X : [v, x - x_0] < 1\}. \tag{13}$$

Note that

$$A_u^+ \cap A_u^- = \emptyset \qquad \text{and} \qquad A_u^+ \cup A_u^- = \mathbb{R}^n \times X$$

and

$$X_{v,x_0}^+ \cap X_{v,x_0}^- = \emptyset \qquad \text{and} \qquad X_{v,x_0}^+ \cup X_{v,x_0}^- = X.$$

If $(v, x_0) \in A_u^+$ then $X_{v,x_0}^+ \neq \emptyset$, indeed, $u \in X_{v,x_0}^+$. The same argument shows that $X_{v,x_0}^- \neq \emptyset$ for $(v, x_0) \in A_u^-$.

Theorem 1 *Let γ defined by (10). Then*

$$\gamma = \sup_{(v,x_0) \in A_u^+} \frac{1}{\mu(X_{v,x_0}^+)}. \tag{14}$$

Proof: Each function $l \in L$ is defined by (8), where $v \in \mathbb{R}^n, x_0 \in X$ and $c, d \in \mathbb{R}_+$ with $c \geq d$. If l defined by (8) then

$$\int_X l(x)d\mu = c\mu(X_{v,x_0}^+) + d\mu(X_{v,x_0}^-) \tag{15}$$

We have also $l(u) = c$ if $(v, x_0) \in A_u^+$ and $l(u) = d$ if $(v, x_0) \in A_u^-$. We can present $L(X)$ as the union of two disjoint sets. One of them contains functions l of the form (8) with $(v, x_0) \in A_u^+$ and arbitrary $c \geq d \geq 0$. The other contains functions l with $(v, x_0) \in A_u^-$ and arbitrary $c \geq d \geq 0$. Applying (10) and (15) we conclude that $\gamma = \max(\gamma^+, \gamma_-)$, where

$$\gamma^+ = \sup_{(v,x_0) \in A_u^+,\, c \geq d \geq 0} \frac{c}{c\mu(X_{v,x_0}^+) + d\mu(X_{v,x_0}^-)} \tag{16}$$

$$\gamma^- = \sup_{(v,x_0) \in A_u^-,\, c \geq d \geq 0} \frac{d}{c\mu(X_{v,x_0}^+) + d\mu(X_{v,x_0}^-)} \tag{17}$$

It follows from (16) and (17) respectively, that

$$\gamma^+ = \sup_{(v,x_0)\in A_u^+,\, 0\le c'\le 1} \frac{1}{\mu(X^+_{v,x_0}) + c'\mu(X^-_{v,x_0})} = \sup_{(v,x_0)\in A_u^+} \frac{1}{\mu(X^+_{v,x_0})},$$

$$\gamma^- = \sup_{(v,x_0)\in A_u^-,\, c'\ge 1} \frac{1}{c'\mu(X^+_{v,x_0}) + \mu(X^-_{v,x_0})} = \sup_{(v,x_0)\in A_u^-} \frac{1}{\mu(X^+_{v,x_0}) + \mu(X^-_{v,x_0})} = \frac{1}{\mu(X)}.$$

Since $\mu(X^+_{v,x_0}) \le \mu(X)$ for all (v, x_0) we conclude that $\gamma = \max(\gamma^+, \gamma^-) = \gamma^+$. △

Corollary 1 Let $u \in X$ and γ is defined by (14). Then

$$f(u) \le \gamma \int_X f(x)d\mu \tag{18}$$

for each $f \in Q^+(X)$ and the constant γ is asymptotically sharp.

Corollary 2 If $\inf_{(v,x_0)\in A_u^+} \mu(X^+_{v,x_0})$ is attained, then the constant γ in the inequality (18) is sharp.

Proof: Let $\inf_{(v,x_0)\in A_u^+} \mu(X^+_{v,x_0}) = \mu(X^+_{w,z_0})$ where $(w, z_0) \in A_u^+$. Let

$$l(x) = \begin{cases} 1 & \text{if } \; x \in X^+_{v,x_0} \\ 0 & \text{if } \; X^-_{v,x_0}. \end{cases}$$

Since $(w, z_0) \in A_u^+$ it follows that $u \in X^+_{v,x_0}$ so $l(u) = 1$. We have also

$$\gamma = \frac{1}{\inf_{(v,x_0)\in A_u^+} \mu(X^+_{v,x_0})} = \frac{1}{\mu(X^+_{w,z_0})}, \qquad \int_X l(x)d\mu = \mu(X^+_{w,z_0}).$$

Thus $l(u) = \gamma \int_X l(x)d\mu$. △

4 Hadamard type inequality for quasiconvex functions vanishing at zero

Let $X \subset \mathbb{R}^n$ be a closed convex set such that $0 \in X$. Let $Q_0^+(X)$ be the set of all evenly quasiconvex nonnegative functions defined on X and such that $f(0) = 0$ and let $L_0(X)$ be the set of restrictions to X of functions from L_0. In other words, $l \in L_0(X)$ if there exists $v \in \mathbb{R}^n$ and a number $c > 0$ such that $l(x) = l^{v,c}(x)$, $(x \in X)$, where $l^{v,c}$ is defined by (7). It follows from Proposition 2 that $L_0(X)$ is a supremal generator of $Q_0(X)$.

Consider an increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(x) > 0$ for all $x > 0$ and

$$\lambda_h = \sup_{c>0} \frac{c}{h(c)} < +\infty \tag{19}$$

and let

$$\psi_h(f) = \int_X h(f(x))d\mu, \qquad f \in Q_0(X), \tag{20}$$

where μ is a finite Borel measure on X. Clearly ψ_h is an increasing functional defined on $Q_0(X)$. Let $u \in X$ and

$$\gamma_h = \sup_{l\in L_0(X)} \frac{l(u)}{\int_X h(l(x))d\mu}. \tag{21}$$

Let $u \in X$. Consider sets

$$B_u = \{v \in \mathbb{R}^n : [v,u] \geq 1\} \qquad \text{and} \qquad X_v = \{x \in X : [v,x] \geq 1\}, \quad (v \in B_v).$$

Note that B_u is not empty for all $u \neq 0$ and X_v is not empty for all $u \neq 0$, since $u \in X_v$. Let

$$\gamma_* = \sup_{v \in B_u} \frac{1}{\mu(X_v)} \tag{22}$$

Theorem 2 *We have* $\gamma_h = \lambda_h \cdot \gamma_*$

Proof: Let $l \in L_0(X)$. Then there exists $v \in \mathbb{R}^n$ and $c \geq 0$ such that $l(x) = l^{v,c}(x)$ for $x \in X$. We have

$$\gamma_h = \sup_{v \in \mathbb{R}^n,\, c \geq 0} \frac{l^{v,c}(u)}{\int_X h(l^{v,c}(x)) d\mu}.$$

Since $l^{v,c}(u) = 0$ if either $v \notin B_u$ or $c = 0$, it follows that

$$\gamma_h(u) = \sup_{v \in B_u,\, c>0} \frac{c}{\int_X h(c) d\mu} = \sup_{v \in B_u,\, c>0} \frac{c}{h(c)} \frac{1}{\mu(X_v)} = \sup_{c>0} \frac{c}{h(c)} \sup_{v \in B_u} \frac{1}{\mu(X_v)} = \lambda_h \cdot \gamma_*.$$

△

Corollary 3 Let $u \in X$ and let λ_h and γ_* are defined by (21) and (22), respectively and let $\gamma_h = \lambda_h \cdot \gamma_*$. Then

$$f(u) \leq \gamma_h \int_X f(x) d\mu \tag{23}$$

for each $f \in Q_0^+(X)$ and the constant $\gamma_h = \lambda_h \gamma_*$ is asymptotically sharp.

Corollary 4 If $\inf_{(v,x_0) \in B_u} \mu(X_v)$ is attained and $\sup_{c>0}(c/h(c))$ is attained then the constant γ_h in the inequality (23 is sharp.

The proof of this Corollary is similar to that of Corollary 2 and we omit it.

It follows from Corollary 3 that the composition $h \circ f$ leads to the presentation of the asymptotically sharp constant γ_h as the product of λ_h and γ_*; the latter corresponds to ψ_I with the identical function I.

Example 1 Let $p > 1$ and

$$\varphi(f) = \int_X (1 + f^p(x)) d\mu, \qquad (f \in Q_0(X).$$

Clearly $\varphi(f) = \psi_h(f)$ where $h(c) = 1 + c^p$. An easy calculation shows that

$$\lambda_h = \sup_{c>0} \frac{c}{1+c^p} = \frac{1}{p}(p-1)^{\frac{1}{p'}},$$

where p' is the conjugate to p factor: $p^{-1} + (p')^{-1} = 1$. Let $u \in X$. Then for each $f \in Q_0(X)$ we have the following asymptotically sharp inequality

$$f(u) \leq \frac{1}{p}(p-1)^{\frac{1}{p'}} \sup_{v \in B_u} \frac{1}{\mu(X_v)} \int_X (1 + f(x)^p) d\mu.$$

5 Examples

In this section we provide an example in which we explicitly obtain the constant γ_* defined by (22) for a unite square $X = [0,1] \times [0,1]$ on the plain $\mathbb{R}^2$. We assume that the Lebesgue measure μ is considered. Recall that

$$\gamma_* = \sup_{v \in B_u} \frac{1}{\mu(X(v))}$$

where where $B_u = \{v \in \mathbb{R}^2 : v_1 u_1 + v_2 u_2 \geq 1\}$ and $X(v) = \{x = (x_1, x_2) \in X : v_1 x_1 + v_2 x_2 \geq 1\}$. Since μ is the Lebesgue measure, $\mu(X(v))$ denotes the area of the set $X(v)$. In order to calculate γ_* it is in fact sufficient to calculate the following

$$\Delta = \inf_{v \in B_u} \mu(X(v))$$

Theorem 3 *Consider the square $X = [-1,1] \times [-1,1]$ in the plane R^2. For any point $(u_1, u_2) \in$ $int X \setminus \{0\}$ we have*

$$\gamma_* = \frac{1}{2(1-|u_1|)(1-|u_2|)}$$

Proof: Case 1 : We will first consider the points $u = (u_1, u_2)$ which lies in $\operatorname{int} \mathbb{R}^2_+$. By simple geometrical arguments we conclude that the straight lines passing through (u_1, u_2) and of the form $v_1 x_1 + v_2 x_2 = 1$ which are required to evaluate Δ are the ones which have $v_1 \geq 0$ and $v_2 \geq 0$. Among these lines the ones that intersect any of the two pairs of parallel sides can again be ruled out by simple geometrical arguments. Hence the only lines passing through (u_1, u_2) which are of interest to us are the lines that intersect the adjacent sides. Considering the fact that $(u_1, u_2) \in \operatorname{int} \mathbb{R}^2_+$ the lines that will be important to us will be the lines of the form $v_1 x_1 + v_2 x_2 = 1$, with $v_1 \geq 0$ and $v_2 \geq 0$ and intersecting the adjacent sides given by the equations $x_1 = 1$ and $x_2 = 1$. So we need basically to calculate the areas of the triangle thus formed by these lines intersecting the adjacent side and then find the minimum among them which will be precisely Δ.

Consider the line given in the form $v_1 x_1 + v_2 x_2 = 1$, where $v_1 \geq 0$ and $v_2 \geq 0$, passing through (u_1, u_2) and intersecting the sides $x_2 = 1$ and $x_1 = 1$ in the points B and C respectively. Let us denote the point $(1,1)$ be A. We need to calculate the area of the triangle ABC. We first calculate the lengths of the segments AB and AC. These are given as

$$AC = \frac{v_1 + v_2 - 1}{v_2} \tag{24}$$

and

$$AB = \frac{v_1 + v_2 - 1}{v_1} \tag{25}$$

Hence Δ can be written precisely as following,

$$\Delta = 0.5 \inf_{v_1, v_2} \frac{(v_1 + v_2 - 1)^2}{v_1 v_2} \tag{26}$$

This minimization has to be carried out under the constrains $v_1 \geq 0$,$v_2 \geq 0$ and that $v_1 u_1 + v_2 u_2 = 1$. This problem can be further reduced to a minimization problem in one variable say v_2 to be carried out over the non-negative part of real line. Hence Δ can be re-expressed as

$$\Delta = 0.5 \inf_{v_2 \geq 0} \frac{(\frac{1 - v_2 u_2}{u_1} + v_2 - 1)^2}{\frac{1 - v_2 u_2}{u_1} v_2} = 0.5 \inf_{v_2 \geq 0} \frac{[(1-u_1) + v_2(u_1 - u_2)]^2}{u_1 v_2 (1 - v_2 u_2)} \tag{27}$$

This function $\dfrac{[(1-u_1)+v_2(u_1-u_2)]^2}{u_1v_2(1-v_2u_2)}$ is not defined for the case $v_2u_2=1$ and $v_2=0$. Now if $v_2u_2=1$, then from $v_1u_1+v_2u_2=1$ we have that $v_1u_1=0$. Since $u_1\neq 0$ we have that $v_1=0$. We also have $v_2=\dfrac{1}{u_2}$. This gives us $\dfrac{1}{u_2}x_2=1$. Hence this shows $x_2=u_2$, which is naturally a straight line parallel to the horizontal axis and passing through (u_1,u_2). This line parallel to the horizontal axis naturally intersects the pair of parallel sides of the unit square which are parallel to the vertical axis and is not of interest to us. The case $v_2=0$ naturally generates a line parallel to the vertical axis and passing through (u_1,u_2). This case as before is not interesting to us. Now for the remaining valued of v_2 we see that function is differentiable. Before calculating the derivative it will be interesting to determine the nature of the function $f(v_2)=\dfrac{[(1-u_1)+v_2(u_1-u_2)]^2}{u_1v_2(1-v_2u_2)}$. We observe that the function has an infinite discontinuity at at $v_2=\dfrac{1}{u_2}$ and $f(v_2)\to+\infty$ as $v_2\to 0+$ and $f(v_2)\to-\frac{(u_1-u_2)^2}{u_2}$ as $v_2\to\infty$. Hence the minimum is achieved at a point $v<\dfrac{1}{u_2}$.
Now we consider $v_2>0$. By taking the derivative and equating it to zero we have that either

$$(1-u_2v_2+u_1v_2-u_1)=0 \tag{28}$$

or

$$2(u_1-u_2)(v_2-u_2v_2^2)=(1-u_2v_2+u_1v_2-u_1)(1-2u_2v_2). \tag{29}$$

Since $u_1\neq 1$ and $(u_1,u_2)\in \operatorname{int}\mathbb{R}^2_+$ we see that the first of these two equations is not valid when $u_1=u_2$. If $u_1\neq u_2$, then we have due to (28):

$$v_2=\frac{u_1-1}{u_1-u_2}. \tag{30}$$

Now from (29) we have

$$v_2=\frac{u_1-1}{2u_1u_2-u_1-u_2}. \tag{31}$$

Now as both $u_1>0$ and $u_2>0$ and $v_2>0$ we see that (31) is valid if $u_2>u_1$. Again as the solution has to be strictly lesser than $\dfrac{1}{u_2}$ a solution of the form (30) with $u_2>u_1$ will lead us to the fact that $u_1>1$, which is surely a contradiction. Hence we need to consider only the solution of the form (31). The expression can be now given for γ_* by plugging in the value of v_2 given by (31) in the expression $f(v_2)$ and observing that $\Delta=0.5f(v_2)$ we have

$$\gamma_*=\frac{1}{2(1-u_1)(1-u_2)}. \tag{32}$$

Case 2. We now consider the case when either $u_1=0$ or $u_2=0$ but not both. Assume without loss of generality that $u_1=0$ and $u_2\neq 0$, i.e $u_2>0$. There can be two symmetrical cases. We can consider lines of the form $v_1x+v_2y=1$, with $v_1\geq 0$, and $v_2\geq 0$ or we can consider the lines with $v_1x+v_2y=1$ with $v_1\leq 0$ and $v_2\geq 0$. We will have the same result in both cases and hence we consider the case where $v_1\geq 0$ and $v_2\geq 0$. Proceeding as in the Case 1 we see that we in effect have to minimize the function

$$f(v_2)=\frac{v_2u_2+1-u_2}{v_2u_2} \tag{33}$$

Now for $v_2=0$ we get the line passing through $(0,u_1)$ and is parallel to the horizontal axis. In fact we can demonstrate the existence of a straight line $v_1x+v_2y=1$ with $v_1>0$ and $v_2>0$ which has the same area enclosed above it and bounded by the lines $y = 1$,$x = 1$ and $x = -1$ as that above the line $y = u_2$ (the line through $(0, u_2)$and parallel to the horizontal axis) and bounded by the lines $y = 1$,$x = 1$ and $x = -1$. To construct this line we have to have to join the point $(0, u_2)$ with the point $(-1, 1))$ and produce it to meet at the unit square at the side given by $x = 1$. This is natural since the point $(0, u_2)$ lies on the vertical axis between $(0, 0)$ and $(0, 1)$. Hence we can disregard the case $v_2 = 0$ and thus by taking the derivative of $f(v_2)$ and equating it to zero we arrive at the value of γ_* as

$$\gamma_* = \frac{1}{2(1 - u_2)}$$

Hence in a symmetric fashion if $u_1 > 0$ and $u_2 = 0$ we have

$$\gamma_* = \frac{1}{2(1 - u_1)}$$

Case 3 Due to the symmetric nature of X for points lying in the other three quadrants can be calculated in a similar manner and with sign changes for v_1 and v_2. For example in the interior of the third quadrant we need $v_1 < 0$ and $v_2 < 0$. Due to the similar nature of the calculations we skip them. $\triangle$

Remark 1 After the interior points we now move onto the points in the boundary. Let u be a boundary point of the square X. Then there exists a point $v \in B_u$ such that $\mu(X_v) = 0$, so (see 22) $\gamma_* = +\infty$. Hence this point is not of interest to us. Now we consider the case $(u_1, u_2) = (0, 0)$. Observe that since $f \in Q_0^+$, it follows that $0 = f(0) \le f(x))$. Hence $f(x) \le \gamma' \int_X f(x)dx$, with any $\gamma' \ge 0$. Hence $\gamma_* = 0$ gives us the asymptotically sharp constant. Let us observe from 22 that our calculation of the constant γ_* depends on the fact that the set $B_u \ne \emptyset$. But as $B_u = \emptyset$, when $u = 0$ the above method of calculation , i.e the proof of Theorem is not applicable to the case $(0, 0)$. Note that $(0, 0)$ is the centre of symmetry of the square X and the Hadamard inequality in its classical form is usually studied exactly with respect to the centre of the set.

Example 2 We now give simple numerical examples. If we choose $u_1 = 0.5$ and $u_2 = 0.5$ from (32) we get $\gamma_* = 2$. For more general case where $u_1 = u_2 = c$ say we have $\gamma_* = \frac{1}{2(1-c)^2}$. If $(u_1, u_2) = (0.5, 0.25)$ we have $\gamma_* = \frac{4}{3}$. Note that γ_* tends to 0.5, as u tends to zero and γ_* tends to $+\infty$ as u tends to a boundary point of the square X.

We now examine the simplest case $n = 1$.

Proposition 4 *Consider $n = 1$ and $X = [-1, +1]$, then*

$$\gamma = \gamma_* = \begin{cases} \frac{1}{\min(1+u, 1-u)} & u \in (-1, 1),\ u \ne 0 \\ \gamma = 1, \gamma_* = 0 & u = 0, \end{cases}$$

where γ and γ_* are defined by (10) and (22), respectively.

Proof: First, recall that an arbitrary quasiconvex function on the real line is evenly quasiconvex.

In the introduction to this article we have seen that for a nonnegative quasiconvex function f defined on $[0, 1]$ the following inequality holds (see [4] for details):

$$f(u) \le \frac{1}{\min(u, 1 - u)} \int_0^1 f(x)dx.$$

It is easy to check that the constant γ' in this inequality is sharp.

This result can be easily extended for an arbitrary segment $[a, b]$. Using the transformation $u = \dfrac{v-a}{b-a}$, $v \in (a, b)$ we can easily check that for an arbitrary nonnegative quasiconvex function f defined on the segment $[a, b]$ of the real line and an arbitrary point $v \in (a, b)$ the following inequality holds:

$$f(v) \leq \frac{1}{\min(v-a, b-v)} \int_a^b f(y)dy$$

and the constant

$$\gamma = \frac{1}{\min(v-a, b-v)}$$

is sharp on the class of all such functions. Thus, considering $X = [-1, +1]$ we have for any $u \in (-1, 1)$

$$\gamma = \frac{1}{\min(1+u, 1-u)}$$

Now using the method demonstrated in Theorem 3, for nonnegative quasiconvex functions with $u \in \operatorname{int} X \setminus \{0\}$ we have

$$\gamma_* = \frac{1}{\min(1+u, 1-u)}$$

Hence $\gamma = \gamma_*$ for $u \in \operatorname{int} X \setminus \{0\}$. We have also that $\gamma = 1$ and $\gamma_* = 0$ for $u = 0$.

Remark 2 Let us observe that there is an interesting similarity in the structure of γ_* for the cases $n = 1$ and $n = 2$.

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References

[1] S. S. Dragomir, J. E. Pečarić and L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* **21** (1995), 335-341.

[2] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[3] S. S. Kutateladze and A. M. Rubinov, Minkowski duality and its applications, *Russian Mathematical Surveys* **27** (1972), 137-192.

[4] C.E.M. Pearce and A.M. Rubinov, P-functions, quasiconvex functions and Hadamard-type inequalities, Journal of Mathematical Analysis and Applications, **240**(1999), 92 -104.

[5] A.M. Rubinov *Abstract convexity and global optimization*, Kluwer Academic Publishers, Dordrecht, 2000.

[6] A. M. Rubinov and B. M. Glover, On generalized quasiconvex conjugation, *Contemporary Mathematics* **204**, 199-216.