

# INTERPOLATIONS OF JENSEN'S INTEGRAL INEQUALITY

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ABSTRACT. Weighted and unweighted interpolations of general order are given for Jensen's integral inequality. Various upper-bound estimates are made for the differences between the interpolates and some convergence results derived. The results generalise and subsume a body of earlier work and employ streamlined proofs.

## 1. INTRODUCTION

A central tool in the applied literature is Jensen's weighted integral inequality, the basic form of which is as follows.

**Theorem 1.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be measurable and denote by  $I$  the convex hull of the image of  $[a, b]$  under  $f$ . Let  $\phi : I \rightarrow \mathbb{R}$  be convex and suppose that  $g$ ,  $fg$  and  $(\phi \circ f) \cdot g$  are all integrable on  $[a, b]$ . If  $g(t) \geq 0$  on  $[a, b]$  and  $\int_a^b g(t) dt > 0$ , then*

$$(1.1) \quad \phi \left( \frac{\int_a^b f(t) g(t) dt}{\int_a^b g(t) dt} \right) \leq \frac{\int_a^b (\phi \circ f)(t) g(t) dt}{\int_a^b g(t) dt}.$$

A convenient standardisation is suggested by the ubiquitous applications of Jensen's inequality in probability. If we define

$$p(t) := g(t) / \int_a^b g(t) dt,$$

then  $p$  is nonnegative and satisfies  $\int_a^b p(t) dt = 1$  and so may be regarded as a probability density function on  $[a, b]$ . With this notation, (1.1) takes the simple form

$$(1.2) \quad \phi \left( \int_a^b f(t) p(t) dt \right) \leq \int_a^b (\phi \circ f)(t) p(t) dt.$$

Without loss of generality we may work with this simpler canonical form.

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Recently Dragomir and Goh [10] derived an estimate for the difference between the two sides of a multivariate version of (1.1). In our present notation, the univariate case of their estimate is

$$(1.3) \quad \begin{aligned} 0 &\leq \int_a^b (\phi \circ f)(t) p(t) dt - \phi \left( \int_a^b f(t) p(t) dt \right) \\ &\leq \int_a^b (\phi' \circ f)(t) \cdot f(t) p(t) dt \\ &\quad - \int_a^b (\phi' \circ f)(t) p(t) dt \cdot \int_a^b f(t) p(t) dt, \end{aligned}$$

provided that all the integrals exist and  $\phi$  is differentiable convex on  $\mathbb{R}$ .

In this paper we give some refinements of these results. For notational convenience, we introduce the  $k$ -variate linear integral operator

$$I_k \{ \cdot \} := \int_a^b \dots \int_a^b (\cdot) p(t_1) \dots p(t_k) dt_1 \dots dt_k.$$

In this notation, (1.2) now becomes

$$(1.4) \quad \phi(I_1 \{f(t)\}) \leq I_1 \{(\phi \circ f)(t)\}$$

and (1.3) reads

$$(1.5) \quad \begin{aligned} 0 &\leq I_1 \{(\phi \circ f)(t)\} - \phi(I_1 \{f(t)\}) \\ &\leq I_1 \{(\phi' \circ f)(t) \cdot f(t)\} - I_1 \{(\phi' \circ f)(t)\} \cdot I_1 \{f(t)\}. \end{aligned}$$

In Section 2 we interpolate (1.4), using both weighted and unweighted (that is, uniformly weighted) forms. The  $k$ -th order weighted and unweighted interpolates are respectively

$$\varphi_k(u) := I_k \left\{ \phi \left( \sum_{i=1}^k u_i f(t_i) \right) \right\}$$

and

$$\varphi_k := I_k \left\{ \phi \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\}.$$

Here  $u = (u_1, \dots, u_k)$  is a set of probability weights, that is, each  $u_i \geq 0$  and  $\sum_{i=1}^k u_i = 1$ , and it is envisaged that  $k$  is a fixed positive integer. When we wish to vary the order  $k$  the extended notation  $u^{(k)} = (u_{1,k}, \dots, u_{k,k})$  will be used.

The basic result is Theorem 2, which generalises a number of known results. We shall see that the  $k$ -th order weighted interpolate  $\varphi_k(u)$  is minimised by the unweighted interpolate  $\varphi_k$ , that is, when each  $u_i = 1/k$ . In Section 3 we give upper bounds for the difference between the first and third terms in (2.1) below. By virtue of the noted minimisation result, our estimates include as a special case an upper bound for the difference between the first and second terms in (2.1). A convergence theorem is established for the difference with  $k \rightarrow \infty$ .

In Section 4 we treat the sequence  $(\varphi_k(u^{(k)}) - \varphi_k)_{k \geq 1}$ . Some results for the sequence  $(\varphi_k - \varphi_{k+1})_{k \geq 1}$  are deduced in Section 5. We conclude in Section six with some remarks on applications to Hadamard's inequalities.

Our arguments exploit the standardisation of  $p$  being a probability density. Suppose  $Y_1, \dots, Y_k$  are independent random variables with common density function  $p$  and define  $X_1, \dots, X_k$  by  $X_i = f(Y_i)$  ( $i = 1, \dots, k$ ). We shall also write  $X, Y$

for a generic pair  $X_i, Y_i$ . Then  $I_k$  is simply the expectation operator with respect to the minimal completed sigma field  $(\mathbf{F}_k)$  generated by  $Y_1, \dots, Y_k$ . Denoting the mean of  $X_1$  by  $E(X_1)$ , as is customary, we then have  $E(X_1) = I_1\{f(t_1)\}$ . Since  $(\mathbf{F}_1)$  is a sub sigma field of  $(\mathbf{F}_k)$ , we have also  $E(X_1) = I_k\{f(t_1)\}$ . We may now express (1.2), (1.5) slightly more succinctly and considerably more evocatively as respectively

$$\phi(E(X)) \leq E(\phi(X))$$

and

$$0 \leq E(\phi(X)) - \phi(E(X)) \leq E(X\phi'(X)) - E(\phi'(X))E(X).$$

We shall lean heavily on this probabilistic formulation both for compact notation within our proofs and for streamlining the algebra involved in them. The assumptions of Theorem 1 are presumed throughout without further comment and with the standardisation that  $g$  is replaced by a probability density function  $p$ . A number of useful bounds arise *via* the Cauchy–Schwarz inequality. In each such connection we shall assume in addition without further comment that  $f^2$  is integrable, and introduce

$$\sigma := \left[ I_1 \{f^2(t)\} - (I_1 \{f(t)\})^2 \right]^{1/2}.$$

Probabilistically this states that

$$\sigma^2 = E(X^2) - [E(X)]^2 =: \text{var}(X),$$

the variance of  $X$ . The basic probabilistic results we shall invoke are that  $E(U^2) = \text{var}(U)$  when  $E(U) = 0$  and that for independent random variables  $X_1, \dots, X_k$  and constants  $u_1, \dots, u_k$ , we have

$$\text{var} \left( \sum_{i=1}^k u_i X_i \right) = \sum_{i=1}^k u_i^2 \text{var}(X_i).$$

For notation convenience, it will be convenient to introduce into our discussion the auxiliary random variables

$$Z_1 = Z_{1,k} := \sum_{i=1}^k u_i X_i$$

and

$$W_k := \frac{1}{k} \sum_{i=1}^k X_i.$$

It is immediate that  $E(Z_1) = E(W_k) = E(X)$  and that  $\varphi_k(u) = E(\phi(Z_1))$  and  $\varphi_k = E(\phi(W_k))$ .

## 2. BASIC RESULTS

Our first result relates expectations involving weighted and unweighted interpolates and refines (1.1).

**Theorem 2.** *For each  $k \geq 1$  and set of probability weights  $u^{(k)}$ , we have*

$$(2.1) \quad \phi(I_1 \{f(t)\}) \leq \varphi_k \leq \varphi_k(u^{(k)}) \leq I_1 \{(\phi \circ f)(t)\}.$$

*Proof.* In probabilistic terms, the result to be proved is that

$$(2.2) \quad \phi(E(X)) \leq E(\phi(W_k)) \leq E(\phi(Z_1)) \leq E(\phi(X)).$$

By Jensen's integral inequality we have

$$E\{\phi(W_k)\} \geq \phi(E\{W_k\}) = \phi(E(X)),$$

the first inequality in the enunciation.

For fixed  $k$  put  $X_{i+k} := X_i$  and for  $1 \leq j \leq k$  define

$$Z_j := \sum_{i=1}^k u_i X_{i+j-1},$$

which is consistent with the definition of  $Z_1$ . Then  $E(Z_j) = E(X)$  and  $E(\phi(Z_j)) = E(\phi(Z_1))$ .

By Jensen's discrete inequality we have

$$\phi\left(\frac{1}{k} \sum_{i=1}^k Z_i\right) \leq \frac{1}{k} \sum_{i=1}^k \phi(Z_i),$$

and since  $\sum_{i=1}^k Z_i = kW_k$ , we derive

$$\phi(W_k) \leq \frac{1}{k} \sum_{j=1}^k \phi(Z_j).$$

Taking expectations provides

$$(2.3) \quad E\{\phi(W_k)\} \leq \frac{1}{k} E\left\{\sum_{j=1}^k \phi(Z_j)\right\} = E\{\phi(Z_1)\}.$$

This gives the second inequality in the enunciation.

Finally, by Jensen's discrete inequality again, we have

$$\phi(Z_1) \leq \sum_{i=1}^k u_i \phi(X_i).$$

Taking expectations provides the final desired inequality.  $\square$

If we choose  $u_{i,k+1} = 1/k$  for  $1 \leq i \leq k$  and  $u_{k+1,k+1} = 0$ , then  $\varphi_{k+1}(u)$  becomes  $\varphi_k$ . Thus we have  $\varphi_{k+1} \leq \varphi_k$  and so  $(\varphi_k)_{k \geq 1}$  is a nonincreasing sequence. We have also  $\varphi_1 = E(\phi(X))$ , of course.

The choices  $f(t) := t$  and  $p(t) := 1/(b-a)$  provide the following interpolation of the Hadamard integral inequalities, which we exhibit *in extenso*.

**Corollary 1.** *Suppose  $\phi$  is convex on  $[a, b]$  and that  $u_i$  ( $1 \leq i \leq k$ ) is a set of probability weights. Then*

$$(2.4) \quad \begin{aligned} \phi\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^k} \int_a^b \cdots \int_a^b \phi\left(\frac{1}{k} \sum_{i=1}^k t_i\right) dt_1 \cdots dt_k \\ &\leq \frac{1}{(b-a)^k} \int_a^b \cdots \int_a^b \phi\left(\sum_{i=1}^k u_i t_i\right) dt_1 \cdots dt_k \\ &\leq \frac{1}{b-a} \int_a^b \phi(t) dt. \end{aligned}$$

This subsumes several known results: the first inequality was proved in [11], the second in [8] and the last in [4]. We pick up these threads again in Section 6.

### 3. BOUNDS FOR THE DIFFERENCE $\varphi_k(u) - \phi(I_1\{f(t)\})$

**Theorem 3.** Denote by  $\phi'_+$  the right derivative of  $\phi$  on the interior  $\mathring{I}$  of  $I$ . Then

$$(3.1) \quad \begin{aligned} 0 &\leq \varphi_k(u) - \phi(I_1\{f(t)\}) \\ &\leq I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \sum_{j=1}^k u_j f(t_j) \right\} \\ &\quad - I_1\{f(t)\} \cdot I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \right\}. \end{aligned}$$

*Proof.* We already have the first inequality and wish to prove the second. We may express (3.1) probabilistically as

$$0 \leq E\{\phi Z_1\} - E(X) \leq E\{Z_1 \phi'_+\} - E(X) \cdot E\{\phi'_+\}.$$

Since  $\phi$  is convex on  $I$ ,

$$\phi(x) - \phi(y) \geq \phi'_+(y)(x - y) \text{ for all } x, y \in \mathring{I}$$

and  $\phi'_+(\cdot)$  is nonnegative on  $\mathring{I}$ . Taking  $x = E(X)$  and  $y = Z_1$ , we deduce that

$$\phi(E(X)) - \phi(Z_1) \geq \phi'_+(Z_1)[E(X) - Z_1].$$

Taking expectations yields

$$(3.2) \quad E\{\phi(Z_1)\} - \phi(E(Z_1)) \leq E\{Z_1 \phi'_+(Z_1)\} - E\{Z_1\} \cdot E\{\phi'_+(Z_1)\},$$

whence we have the desired result.  $\square$

When each  $u_i = 1/k$ , we may exploit symmetry in  $j$  of the summand in (3.1) to simplify the conclusion of the last theorem to

$$(3.3) \quad \begin{aligned} 0 &\leq \varphi_k(u) - \phi(I_1\{f(t)\}) \\ &\leq I_k \left\{ f(t_1) \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\} \\ &\quad - I_1\{f(t)\} \cdot I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\}. \end{aligned}$$

The previous theorem may be extended as follows.

**Theorem 4.** For  $k \geq 1$  we have

$$(3.3) \quad \begin{aligned} 0 &\leq I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \sum_{j=1}^k u_j f(t_j) \right\} \\ &\quad - I_1\{f(t)\} \cdot I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \right\} \\ &\leq \sigma \sqrt{\sum_{j=1}^k u_j^2} \left[ I_k \left\{ \left[ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \right]^2 \right\} \right]^{1/2}. \end{aligned}$$

*Proof.* Since  $E(X) = E(Z_1)$ , the middle term in (3.3) can be cast probabilistically as

$$E \{ \phi'_+(Z_1) \times [Z_1 - E(Z_1)] \},$$

which by the Cauchy–Schwarz inequality is less than or equal to

$$\left\{ E \left\{ [\phi'_+(Z_1)]^2 \right\} \right\}^{1/2} \left\{ E \left\{ [Z_1 - E\{Z_1\}]^2 \right\} \right\}^{1/2}.$$

Further  $E[Z_1 - E(Z_1)] = 0$ , so we have

$$E[Z_1 - E(Z_1)]^2 = \text{var}(Z_1) = \sum_{i=1}^k u_i^2 \text{var}(X_i) = \sigma^2 \sum_{i=1}^k u_i^2,$$

and the desired result follows.  $\square$

As with the previous theorem, (3.3) simplifies when  $u_i = 1/k$  for each  $i$ , becoming

$$\begin{aligned} 0 &\leq I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) f(t_1) \right\} - I_1 \{ f(t) \} \cdot I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\} \\ &\leq \sigma k^{-1/2} \left[ I_k \left\{ \left[ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right]^2 \right\} \right]^{1/2}. \end{aligned}$$

for all  $k \geq 1$ .

**Corollary 2.** *Suppose that*

$$(3.4) \quad M := \sup_{x \in I} |\phi'_+(x)| < \infty$$

and

$$(3.5) \quad \sum_{j=1}^k u_{j,k}^2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Then

$$\varphi_k \left( u^{(k)} \right) \rightarrow \phi \left( I_1 \{ f(t) \} \right) \text{ as } k \rightarrow \infty.$$

We note that the second assumption is automatically satisfied in the particular case  $u_{j,k} = 1/k$  for  $1 \leq j \leq k$ .

The conclusion of the corollary may be expressed probabilistically as

$$E \{ Z_{1,k} \} \rightarrow \phi(E(X)) \text{ as } k \rightarrow \infty.$$

#### 4. BOUNDS FOR $\varphi_k(u) - \varphi_k$

The difference between the outermost terms in (2.1) can be used to provide a crude upper bound for  $\varphi_k(u) - \varphi_k$ . Here we provide tighter bounds.

**Theorem 5.** For  $k \geq 1$  we have

$$\begin{aligned} 0 &\leq \varphi_k(u) - \varphi_k \\ &\leq I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \sum_{j=1}^k u_j f(t_j) \right\} \\ &\quad - I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \frac{1}{k} \sum_{j=1}^k f(t_j) \right\}. \end{aligned}$$

*Proof.* By the convexity of  $\phi$

$$\phi(W_k) - \phi(Z_1) \geq \phi'_+(Z_1)(W_k - Z_1).$$

Taking expectations provides

$$E(\phi(Z_1)) - E(\phi(W_k)) \leq E\{Z_1 \phi'_+(Z_1)\} - E\{W_k \phi'_+(Z_1)\},$$

which is the desired result in probabilistic form.  $\square$

The estimate is continued by the next theorem.

**Theorem 6.** For each  $k \geq 1$ ,

$$\begin{aligned} &I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \sum_{j=1}^k u_j f(t_j) \right\} \\ &\quad - I_k \left\{ \phi'_+ \left( \sum_{i=1}^k u_i f(t_i) \right) \frac{1}{k} \sum_{j=1}^k f(t_j) \right\} \\ &\leq \sigma \sqrt{\sum_{i=1}^k (u_i - 1/k)^2} \times \left[ I_k \left\{ \left[ \phi'_+ \left( \sum_{j=1}^k u_j f(t_j) \right) \right]^2 \right\} \right]^{1/2}. \end{aligned}$$

*Proof.* The left-hand side of this inequality is

$$E\{\phi'_+(Z_1) \times (Z_1 - W_k)\},$$

which by the Cauchy-Schwarz inequality is less than or equal to

$$\left( E\{[\phi'_+(Z_1)]^2\} \right)^{1/2} \times \left( E\{[Z_1 - W_k]^2\} \right)^{1/2}.$$

Since  $E(Z_1 - W_k) = 0$ , we may compute the second term in parentheses as

$$\text{var} \left\{ \sum_{j=1}^k \left( u_j - \frac{1}{k} \right) X_j \right\} = \sum_{j=1}^k \left( u_j - \frac{1}{k} \right)^2 \text{var}(X_j) = \sigma^2 \sum_{j=1}^k \left( u_j - \frac{1}{k} \right)^2,$$

from which we deduce the desired estimate.  $\square$

**Corollary 3.** If (3.4) applies, then

$$0 \leq \varphi_k(u) - \varphi_k \leq M\sigma \left[ \sum_{i=1}^k \left( u_i - \frac{1}{k} \right)^2 \right].$$

It follows that subject to (3.4), a sufficient condition for

$$\lim_{k \rightarrow \infty} \varphi_k(u) = \phi(I_1\{f(t)\})$$

is that

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k (u_{i,k} - 1/k)^2 = 0.$$

This is readily seen to be the same condition as (3.5).

## 5. UPPER BOUNDS FOR $\varphi_k - \varphi_{k+1}$

From (2.2), we have

$$(5.1) \quad \phi(E(X)) \leq \varphi_{k+1} \leq \varphi_k \leq \dots \leq E(\phi(X))$$

for  $k \geq 1$ , so that the difference  $\varphi_k - \varphi_{k+1}$  is nonnegative and can be ascribed a uniform upper bound  $E(\phi(X)) - \phi(E(X))$  which is independent of  $k$ . The next theorem refines this to a tighter and  $k$ -dependent bound.

**Theorem 7.** For each  $k \geq 1$ ,

$$(5.2) \quad \begin{aligned} 0 &\leq \varphi_k - \varphi_{k+1} \\ &\leq \frac{1}{k+1} \left[ I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) f(t_1) \right\} \right. \\ &\quad \left. - I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\} I_1\{f(t)\} \right]. \end{aligned}$$

*Proof.* As  $\phi$  is convex,

$$\begin{aligned} &\phi \left( \frac{1}{k+1} \sum_{i=1}^{k+1} X_i \right) - \phi \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \\ &\geq \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \left( \frac{1}{k+1} \sum_{j=1}^{k+1} X_j - \frac{1}{k} \sum_{j=1}^k X_j \right) \\ &= \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \left[ \frac{X_{k+1}}{k+1} - \frac{1}{k(k+1)} \sum_{j=1}^k X_j \right] \end{aligned}$$

for all  $k \geq 1$ .

Taking expectations provides

$$\begin{aligned} &\varphi_{k+1} - \varphi_k \\ &\geq \frac{1}{k+1} \left[ E \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \right\} E(X) - E \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \frac{1}{k} \sum_{j=1}^k X_j \right\} \right] \\ &= \frac{1}{k+1} \left[ E \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \right\} E(X) - E \left\{ X_1 \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \right\} \right], \end{aligned}$$

where symmetry has been coupled with a change of variables to provide the last step.  $\square$

The above result is continued by the following one.

**Theorem 8.** *For each  $k \geq 1$  we have*

$$\begin{aligned} & \frac{1}{k+1} \left[ I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) f(t_1) \right\} - I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\} I_1 \{f(t)\} \right] \\ & \leq \frac{\sigma}{\sqrt{k(k+1)}} \left[ I_k \left\{ \left[ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right]^2 \right\} \right]^{1/2}. \end{aligned}$$

*Proof.* The left-hand side can be expressed as

$$\frac{1}{k+1} E \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \left[ X_{k+1} - \frac{1}{k} \sum_{j=1}^k X_j \right] \right\},$$

which by the Cauchy-Schwarz inequality is less than or equal to

$$\left( E \left\{ \left[ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^k X_i \right) \right]^2 \right\} \right)^{1/2} \times \left( E \left\{ \left[ X_{k+1} - \frac{1}{k} \sum_{i=1}^k X_i \right]^2 \right\} \right)^{1/2}.$$

Since  $E \left( X_{k+1} - (1/k) \sum_{i=1}^k X_i \right) = 0$ , the expression within the second pair of parentheses is

$$\text{var} \left( X_{k+1} - \frac{1}{k} \sum_{i=1}^k X_i \right) = \text{var}(X_{k+1}) + \frac{1}{k^2} \sum_{i=1}^k \text{var}(X_i) = \frac{(k+1)\sigma^2}{k},$$

from which we deduce the desired result.  $\square$

Finally we have the following corollary.

**Corollary 4.** *If (3.4) holds, then for all  $\alpha \in [0, 1)$  we have*

$$\lim_{n \rightarrow \infty} (\varphi_n - \varphi_{n+1}) n^\alpha = 0.$$

*Proof.* By the two preceding theorems,

$$(5.3) \quad 0 \leq \varphi_n - \varphi_{n+1} \leq \frac{M\sigma}{\sqrt{n(n+1)}}$$

for all  $n \geq 1$ , whence the result.  $\square$

## 6. APPLICATIONS TO HADAMARD'S INEQUALITIES

We conclude by resuming from Corollary 1 and the observations made there. Hadamard's inequality states that if  $\phi : I \rightarrow \mathbb{R}$  is convex on the interval  $I = [a, b]$  of real numbers, then

$$(6.1) \quad \phi \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b \phi(x) dx \leq \frac{\phi(a) + \phi(b)}{2}.$$

Denote by

$$J_k \{ \cdot \} := \frac{1}{(b-a)^k} \int_a^b \dots \int_a^b (\cdot) dx_1 \dots dx_k$$

the special case of  $I_k$  when  $p(x) := 1/(b-a)$  on  $[a, b]$ . Dragomir, Pečarić and Sándor [11] have interpolated the first inequality in (6.1) as

$$(6.2) \quad \phi\left(\frac{a+b}{2}\right) \leq J_{k+1} \left\{ \phi\left(\frac{1}{k+1} \sum_{i=1}^{k+1} x_i\right) \right\} \leq J_k \left\{ \phi\left(\frac{1}{k} \sum_{i=1}^k x_i\right) \right\} \leq \dots \leq J_1 \{ \phi(x) \}$$

for all  $k \geq 1$ . This is a particular case of (5.1).

Dragomir [4] has also established a weighted interpolation, in our notation

$$(6.3) \quad \phi\left(\frac{a+b}{2}\right) \leq J_k \left\{ \phi\left(\sum_{i=1}^k u_i x_i\right) \right\} \leq J_1 \{ \phi(x) \},$$

of Hadamard's first inequality. This was subsequently improved by Dragomir and Buse [8] who proved *inter alia* that

$$(6.4) \quad J_k \left\{ \phi\left(\frac{1}{k} \sum_{i=1}^k x_i\right) \right\} \leq J_k \left\{ \phi\left(\sum_{i=1}^k u_i x_i\right) \right\}.$$

This is Theorem 2 with  $f(x) := x$  (and so  $X_i = Y_i$ ).

From Corollary 2 we can obtain the following result which was derived by a different argument in [9].

Suppose  $\phi : I \rightarrow \mathbb{R}$  is convex, (3.4) holds and that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n v_i^2}{\left(\sum_{i=1}^n v_i\right)^2} = 0.$$

Then if  $V_n := \sum_{i=1}^n v_i > 0$ , we have

$$\lim_{n \rightarrow \infty} J_n \left\{ \phi\left(\sum_{i=1}^n v_i x_i / V_n\right) \right\} = \phi\left(\frac{a+b}{2}\right).$$

Write  $h_n, h_n(u)$  respectively for  $\varphi_n, \varphi_n(u)$  in the case  $p(x) = 1/(b-a)$  on  $[a, b]$ . We have the following.

**Proposition 1.** *Let  $\phi : I \rightarrow \mathbb{R}$  be convex and suppose (3.4) holds. Then for all  $a, b \in I$  with  $a < b$ , we have*

$$0 \leq h_n - h_{n+1} \leq \frac{M(b-a)}{2\sqrt{3}\sqrt{n(n+1)}}$$

for all positive integers  $n$ .

*Proof.* The result is (5.3) with

$$\sigma^2 = \frac{\int_a^b t^2 dt}{b-a} - \left(\frac{\int_a^b t dt}{b-a}\right)^2 = \frac{(b-a)^2}{12}.$$

□

The consequence

$$\lim_{n \rightarrow \infty} [n^\alpha (h_n - h_{n+1})] = 0 \text{ for } \alpha \in [0, 1)$$

is an improvement on the results of [7].

The weighted case is embodied in the following proposition.

**Proposition 2.** *With the assumptions of Proposition 1,*

$$0 \leq h_n(u) - h_n \leq \frac{M(b-a)}{2\sqrt{3}} \left[ \sum_{i=1}^n (u_i - 1/n)^2 \right]^{1/2}$$

for all  $n \geq 1$ .

For other results connected with Hadamard's inequality see [1]–[9], where further references are given.

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