NEW PROOFS FOR INEQUALITIES OF POWER-EXPONENTIAL FUNCTIONS

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ABSTRACT. Some new proofs for the following inequalities of power-exponential functions are given

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x} > \frac{y^x}{x^y}, \quad \left(\frac{y}{x}\right)^{xy} > \frac{y^y}{x^x},$$

where $y > x > 0$ and $(x-1)(y-1) > 0$.

1. INTRODUCTION

Using different techniques, Professor Dr. F. Qi and L. Debnath proved the following in [6]:

Theorem 1. *For $y > x > 0$ and $(x-1)(y-1) > 0$, we have*

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x} > \frac{y^x}{x^y}, \tag{1}$$

$$\left(\frac{y}{x}\right)^{xy} > \frac{y^y}{x^x}. \tag{2}$$

For $0 < x < 1 < y$, the right hand sides of (1) and (2) are reversed.

If $0 < x < 1 < y$ or $0 < x < y < e$, then

$$\frac{y^x}{x^y} > \frac{y \ln x}{x \ln y} \cdot \frac{y^x - 1}{x^y - 1} > 1. \tag{3}$$

If $e < x < y$, then inequality (3) is reversed.

The following lemma is well-known:

Lemma 1 ([3, page 11])**.** *If $f(t)$ is an increasing integrable function on I, then the arithmetic mean of the function $f(t)$,*

$$\Phi(r,s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r \neq s, \\ f(r), & r = s, \end{cases} \tag{4}$$

is also increasing with both r and s on I.

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The above lemma was reproved and used successfully to verify monotonicity, logarithmic convexity, and other properties of the extended mean values, generalized weighted mean values, and generalized abstracted mean values in [4, 5] and [7, 8].

In this paper, using Lemma 1 and other techniques, we will give some new proofs for the inequalities of power-exponential functions presented in Theorem 1.

2. New proofs

2.1. New proof of the right hand side of inequality (1).

2.1.1. *First proof.* Using standard arguments, the right hand side of inequality (1) can be rewritten as

$$(1-x)\ln y > (1-y)\ln x \tag{5}$$

for $y > x > 0$ and $(x-1)(y-1) > 0$. Inequality (5) is equivalent to

$$\frac{\ln y}{1-y} > \frac{\ln x}{1-x}. \tag{6}$$

By Lemma 1, we may deduce that the function ϕ defined by

$$\begin{aligned} \phi(u) &= \frac{\ln u}{u-1} = \frac{1}{u-1}\int_1^u \frac{1}{t}\,\mathrm{d}t, \\ \phi(1) &= 1 \end{aligned} \tag{7}$$

is decreasing for $u > 0$. Therefore, inequality (6) holds and the right hand side of inequality (1) follows.

2.1.2. *Second proof.* We can rewrite the right hand side of inequality (1) as follows.

$$y^{\frac{1}{y-1}} < x^{\frac{1}{x-1}} \tag{8}$$

for $y > x > 0$ and $(x-1)(y-1) > 0$. This is equivalent to the function $t^{\frac{1}{t-1}}$, and then the function $\frac{\ln t}{t-1}$, decreasing in $t > 0$. This was done in [6].

2.2. New proofs of inequality (2).

2.2.1. *First proof.* It is easy to see that inequality (2) is equivalent to

$$\frac{y\ln y}{y-1} > \frac{x\ln x}{x-1} \tag{9}$$

for $y > x > 0$ and $(x-1)(y-1) > 0$.

From Lemma 1, we obtain that the function

$$\begin{aligned} \varphi(u) &= \frac{u\ln u}{u-1} = \frac{1}{u-1}\int_1^u (1+\ln t)\,\mathrm{d}t, \\ \varphi(1) &= 1 \end{aligned} \tag{10}$$

is increasing for $u > 0$. Therefore, inequality (9) holds and (2) follows.

2.2.2. *Second Proof.* Inequality (2) can be rewritten as

$$\frac{(y^y)^x}{(x^x)^y} > \frac{y^y}{x^x}, \tag{11}$$

which is equivalent to

$$y^{\frac{y}{y-1}} > x^{\frac{x}{x-1}} \tag{12}$$

for $y > x > 0$ and $(x-1)(y-1) > 0$. It suffices to prove that the function $\frac{t\ln t}{t-1}$ is increasing in $t > 0$. This was done in [6].

2.3. **New proof of the left hand side of inequality** (1)**.** The equivalent form of the left hand side of inequality (1) is

$$\frac{x^y-1}{y^x-1} > \frac{\ln x}{\ln y} \tag{13}$$

for $(x-1)(y-1)>0$ and $y>x>0$.

Let $\omega(t)=x^{yt}$ and $\psi(t)=y^{xt}$ for $t\in[0,1]$. Direct computation yields

$$\begin{aligned}\omega'(t)&=x^{yt}\cdot y\ln x,\\ \psi'(t)&=y^{xt}\cdot x\ln y.\end{aligned} \tag{14}$$

Then, using the Cauchy's mean-value theorem for differentiation and the right inequality in (1), there exists a point $\xi\in(0,1)$, such that

$$\begin{aligned}\frac{x^y-1}{y^x-1}&=\frac{\omega(1)-\omega(0)}{\psi(1)-\psi(0)}\\ &=\frac{\omega'(\xi)}{\psi'(\xi)}\\ &=\left(\frac{x^y}{y^x}\right)^{\xi}\cdot\frac{y\ln x}{x\ln y}\\ &>\left(\frac{x}{y}\right)^{\xi}\cdot\frac{y\ln x}{x\ln y}\\ &>\left(\frac{y}{x}\right)^{1-\xi}\cdot\frac{\ln x}{\ln y}\\ &>\frac{\ln x}{\ln y}.\end{aligned} \tag{15}$$

2.4. **New proofs of inequality** (3)**.** Since the inequality $x^y<y^x$ holds for $0<x<y<e$ and is reversed for $e<x<y$, (For more information about this inequality, please refer to [1, 2, 3, 6]), from the first three lines in inequality (15), we have that

$$\frac{x^y-1}{y^x-1}=\left(\frac{x^y}{y^x}\right)^{\xi}\cdot\frac{y\ln x}{x\ln y}<\frac{y\ln x}{x\ln y} \tag{16}$$

holds for $0<x<y<e$ and is reversed for $e<x<y$. This implies that the right hand side of inequality (3) holds.

In [6], it was proved that the function $\frac{t-1}{\ln t}$ is increasing in $t>0$. For $y>1>x>0$, we have $y^x>1>x^y$, and thus

$$\frac{y^x-1}{\ln y^x}>\frac{x^y-1}{\ln x^y}. \tag{17}$$

This implies that the right hand side of inequality (3) holds.

For $0<x<y<e$, we have $y^x>x^y$, and applying the same procedure produces an identical result.

It was also proved in [6] that the function $\frac{t-1}{t\ln t}$ decreases in $t>0$. Since the inequality $x^y<y^x$ holds for $0<x<1<y$ and for $0<x<y<e$, then

$$\frac{y^x-1}{y^x\ln y^x}<\frac{x^y-1}{x^y\ln x^y}. \tag{18}$$

Simplification of inequality (18) gives us the left hand side of inequality (3).

Similar arguments enable us to establish the reversed inequalities.

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