A note on two inequalities of Telyakovskii type

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Abstract. New proofs of the Theorems proved by the author in [3] and [4] are given.

1. Introduction

In 1973, S. A. Telyakovskii [2] introduced a Sidon-type condition [1] described by class S in his paper.

A null-sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class S, or briefly $\{a_n\} \in S$ if there exists a monotoniacally decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all n. Telyakovskii [2], firstly proved that the Sidon's class is equivalent to the class S and second that S is a L^1-integrability class for cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\,. \tag{1.1}$$

Thus the class S is known as Sidon-Telyakovskii class S.

Theorem A [2]. *Let the cofficients of the series* (1.1) *belong to the class S. Then the series* (1.1) *is a Fourier series of some $f \in L^1(0,\pi)$ and the following inequality holds:*

$$\int_0^{\pi} |f(x)|dx \leq C \sum_{n=0}^{\infty} A_n, \quad \text{where} \quad C > 0\,.$$

Similar theorem for sine series

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

is also proved for the class S by Telyakovskii [2].

Theorem B [2]. *Let the coefficients of the series* (1.2) *belong to the class S. Then the following relation holds for $p = 1, 2, 3, \ldots$*

$$\int_{\frac{\pi}{p+1}}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n\right)\,.$$

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In particular $g(x)$ *is a Fourier series iff* $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

Very recently, the author of this note [3] defined the following class S_r.

A null sequence $\{a_n\}$ belongs to the class S_r, $r = 0, 1, 2, 3, \ldots$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} n^r A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all n.

When $r = 0$ it is clear that $S_r = S$.

In [3] and [4] we obtained L^1-estimates of the r-th derivatives for the series (1.1) and (1.2), i.e. for the series

$$\sum_{n=1}^{\infty} n^r a_n \cos\left(nx + \frac{r\pi}{2}\right) , \tag{1.3}$$

$$\sum_{n=1}^{\infty} n^r a_n \sin\left(nx + \frac{r\pi}{2}\right) , \tag{1.4}$$

Namely, the following theorems were proved by the author in [3], [4].

Theorem 1 [3]. *Let the coefficients of the series* (1.1) *belongs to the class* S_r, $r = 0, 1, 2, \ldots$. *Then the series* (1.3) *is a Fourier series of some* $f^{(r)} \in L^1(0, \pi)$ *and the following inequality holds:*

$$\int_0^{\pi} |f^{(r)}(x)| dx \leq C \sum_{n=1}^{\infty} n^r A_n\,, \quad C > 0\,.$$

Theorem 2 [4]. *Let the coefficients of the series* $g(x)$ *satisfy the condition* S_r, $r = 0, 1, 2, \ldots$. *Then the following relation holds for* $m = 1, 2, 3 \ldots$.

$$\int_{\frac{\pi}{m+1}}^{\pi} |g^{(r)}(x)| dx = \sum_{n=1}^{m} |a_n|\, n^{r-1} + O\left(\sum_{n=1}^{\infty} n^r A_n\right) .$$

In particular (1.4) is a Fourier series iff $\sum_{n=1}^{\infty} |a_n|\, n^{r-1} < \infty$.

Corollary [4]. *Let the coefficients of the series* $g(x)$ *satisfy the condition* S_r, $r = 1, 2, 3, \ldots$. *Then the following estimate holds:*

$$\int_0^{\pi} |g^{(r)}(x)| dx = O\left(\sum_{n=1}^{\infty} n^r A_n\right) .$$

2. Lemma

For the proofs of the Theorem 1 and Theorem 2, we need the following Lemma.

Lemma 1. [5] *If* $\{a_n\} \in S_r$, $r = 0, 1, 2, 3, \dots$ *then* $\{n^r a_n\} \in S$.

Proof. Let $\{a_n\} \in S_r$. Then

$$n^r a_n = n^r \sum_{k=n}^{\infty} \Delta a_k \le \sum_{k=n}^{\infty} k^r |\Delta a_k| \le \sum_{k=n}^{\infty} k^r A_k = o(1)\,, \quad n \to \infty\,,$$

i.e. $n^r a_n \to 0,\ n \to \infty$.

Now, we consider the sequence $\{B_k\}$ defined as follows

$$B_k = k^r A_k + \sum_{i=k+1}^{\infty} [i^r - (i-1)^r] A_i\,. \tag{1.5}$$

We have:

$$B_k - B_{k+1} = k^r A_k - (k+1)^r A_{k+1} + (k+1)^r A_{k+1} - k^r A_{k+1} = k^r\, \Delta A_k \ge 0\,,$$

i.e. $\{B_k\} \downarrow 0$;

$$\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} k^r A_k + \sum_{k=1}^{\infty}\sum_{i=k}^{\infty}[(i+1)^r - i^r] A_{i+1} =$$

$$= \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty}\sum_{k=1}^{i}[(i+1)^r - i^r] A_{i+1} =$$

$$= \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} i[(i+1)^r - i^r] A_{i+1} <$$

$$< \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty}[(i+1)^{r+1} - i^{r+1}] A_{i+1} \le$$

$$= \sum_{k=1}^{\infty} k^r A_k + O\left(\sum_{i=1}^{\infty} i^r A_i\right) < \infty\,, \quad \text{i.e. } \sum_{k=1}^{\infty} B_k < \infty.$$

Then,

$$\Delta(k^r a_k) = k^r a_k - (k+1)^r a_{k+1} = k^r\, \Delta a_k - \big((k+1)^r - k^r\big) a_{k+1}\,.$$

The function $h(x) = (x+1)^r - x^r$ is monotone decreasing on $[0, \infty)$, since

$$h'(x) = r[(x+1)^{r-1} - x^{r-1}] \ge 0 \ \text{ for } \ x \ge 0.$$

This implies that

$$|\Delta(k^r a_k)| \le k^r |\Delta a_k| + \big((k+1)^r - k^r\big)|a_{k+1}| \le$$

$$\le k^r A_k + \big((k+1)^r - k^r\big) \sum_{i=k+1}^{\infty} |\Delta a_i| \le$$

$$\le k^r A_k + \sum_{i=k+1}^{\infty} \big(i^r - (i-1)^r\big)|\Delta a_i| \le$$

$$\le k^r A_k + \sum_{i=k+1}^{\infty} \big(i^r - (i-1)^r\big)A_i = B_k\,,$$

$|\Delta(k^r a_k)| \le B_k$, for all k. Thus, $\{n^r a_n\} \in S$.

3. Proofs

3.1. Proof of the Theorem 1

By Lemma 1, we have $\{n^r a_n\} \in S$. Applying the Theorem A, we obtain that the series (1.3) is a Fourier series of some $f^{(r)} \in L^1(0,\pi)$ and

$$\int_0^{\pi} |f^{(r)}(x)|dx \le C \sum_{n=0}^{\infty} B_n\,,$$

where $\{B_n\}$ is the sequence defined by (1.5). Hence,

$$\int_0^{\pi} |f^{(r)}(x)|dx \le C \sum_{n=0}^{\infty} n^r A_n + C \sum_{n=0}^{\infty} \sum_{i=n+1}^{\infty} \big(i^r - (i-1)^r\big)A_i =$$

$$= C \sum_{n=0}^{\infty} n^r A_n + O\left(\sum_{i=1}^{\infty} n^r A_i\right) = O\left(\sum_{i=1}^{\infty} i^r A_i\right)\,,$$

i.e. the our inequality is satisfied.

3.2. Proof of the Theorem 2

Applying the Lemma 1 and Theorem B, we obtain

$$\int_{\frac{\pi}{m+1}}^{\pi} |g^{(r)}(x)|dx = \sum_{n=1}^{m} \frac{|n^r a_n|}{n} + O\left(\sum_{n=1}^{\infty} B_n\right)\,,$$

where B_n is the sequence defined by (1.5). Then,

$$\int_{\frac{\pi}{m+1}}^{\pi} |g^{(r)}(x)|dx = \sum_{n=1}^{m} |a_n|\, n^{r-1} + O\left(\sum_{n=1}^{\infty} n^r A_n\right) + O\left(\sum_{k=1}^{\infty} \sum_{i=k+1}^{\infty} \big(i^r - (i-1)^r\big)A_i\right) =$$

$$= \sum_{n=1}^{m} n^{r-1}|a_n| + O\left(\sum_{n=1}^{\infty} n^r A_n\right) + O\left(\sum_{n=1}^{\infty} n^r A_n\right) =$$

$$= \sum_{n=1}^{m} n^{r-1}|a_n| + O\left(\sum_{n=1}^{\infty} n^r A_n\right)\,.$$

3.3. Proof of the Corollary

By inequalities

$$\sum_{n=1}^{m} |a_n|\, n^{r-1} \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} |\Delta a_k| \leq$$

$$\leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} A_k =$$

$$= \sum_{k=1}^{\infty} A_k \sum_{n=1}^{k} n^{r-1} \leq \sum_{k=1}^{\infty} k^r A_k \,,$$

and by Theorem 2, we obtain

$$\int_{\frac{\pi}{m+1}}^{\pi} |g^{(r)}(x)| dx = O\left(\sum_{n=1}^{\infty} n^r A_n \right) .$$

Letting $m \to \infty$, the inequality is satisfied.

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