DIFFERENCES BETWEEN MEANS WITH BOUNDS FROM A RIEMANN-STIELTJES INTEGRAL

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Abstract. An identity for the difference between two integral means is obtained in terms of a Riemann-Stieltjes integral. This enables bounds to be procured when the integrand is of bounded variation, Lipschitzian and monotonic. If f is absolutely continuous, bounds are also obtained for $f' \in L_p[a,b]$, $1 \le p < \infty$, the usual Lebesgue norms. This supplements earlier results involving $f' \in L_\infty[a,b]$.

1. Introduction

The following theorem was proved in Barnett et al. [2].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a,b]$, i.e.,*

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty.$$

Then for $a \le c < d \le b$, we have the inequality

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(u)\,du \right| \\ &\le \left\{ \frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{c+d}{2}}{(b-a)-(d-c)} \right]^2 \right\} [(b-a)-(d-c)] \|f'\|_\infty \\ &\le \frac{1}{2}[(b-a)-(d-c)] \|f'\|_\infty . \end{aligned} \tag{1.1}$$

The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second one.

They utilised the identity

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(u)\,du = \int_a^b K_{c,d}(s) f'(s)\,ds, \tag{1.2}$$

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where

$$K_{c,d}(s) := \begin{cases} \dfrac{a-s}{b-a} & \text{if} \quad s \in [a,c]; \\ \dfrac{s-c}{d-c} + \dfrac{a-s}{b-a} & \text{if} \quad s \in (c,d); \\ \dfrac{b-s}{b-a} & \text{if} \quad s \in [d,b]. \end{cases} \tag{1.3}$$

It was demonstrated that the Ostrowski inequality [12], represented by the following theorem, could be recaptured by using some limiting procedure:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and assume that $|f'(x)| \le M$ for all $x \in (a,b)$. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\, M \tag{1.4}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

For some generalisations and related results, see the book [11, p. 468 - 484], the papers [1] – [12] and the website `http://rgmia.vu.edu.au/` where many papers devoted to this inequality can be accessed on line.

It is the aim of the current article to obtain bounds on

$$D(f;a,c,d,b) := \mathcal{M}(f;a,b) - \mathcal{M}(f;c,d), \quad a \le c < d \le b, \tag{1.5}$$

where

$$\mathcal{M}(f;a,b) := \frac{1}{b-a} \int_a^b f(t)\,dt,$$

in terms of the Lebesgue norms $\|f'\|_p$, $1 \le p < \infty$ with $f' \in L_p[a,b]$ implying $\left(\int_a^b |f'(t)|^p\,dt \right)^{\frac{1}{p}} < \infty$, $1 \le p < \infty$.

Further bounds on $|D(f;a,c,d,b)|$ will be obtained under less restrictive assumptions than absolute continuity on f. Bounds are obtained for f Hölder continuous in Section 2 while in Section 3 bounds are obtained for f of bounded variation, Lipschitzian and monotonic.

2. Results for $f' \in L_p[a,b]$, $1 \le p < \infty$

The following theorem holds (see also Cerone and Dragomir [3]).

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping. Then for $a \le c < d \le b$ the inequalities*

$$|D(f;a,c,d,b)| \le \begin{cases} \frac{b-a}{(q+1)^{\frac{1}{q}}} \left[1 + \left(\frac{\rho}{1-\rho} \right)^q \right]^{\frac{1}{q}} \left[\nu^{q+1} + \lambda^{q+1} \right]^{\frac{1}{q}} \|f'\|_p, \\ \qquad f' \in L_p[a,b], \quad 1 \le p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1; \\ [1 - \rho + |\nu + \rho + \lambda|] \|f'\|_1, \quad f' \in L_1[a,b], \end{cases} \tag{2.1}$$

where $(b-a)\nu = c-a$, $(b-a)\rho = d-c$, $(b-a)\lambda = b-d$.

Proof. From (1.2) and (1.5) we have on using Hölder's integral inequality that

$$|D(f;a,c,d,b)| \leq \left(\int_a^b |K_{c,d}(s)|^q\, dt\right)^{\frac{1}{q}} \|f'\|_p, \quad 1 \leq p < \infty, \ \frac{1}{p}+\frac{1}{q}=1. \tag{2.2}$$

Now,

$$\int_a^c |s-a|^q\, ds = \frac{(c-a)^{q+1}}{q+1} \quad \text{and} \quad \int_d^b |b-s|^q\, ds = \frac{(b-d)^{q+1}}{q+1}. \tag{2.3}$$

Further,

$$\begin{aligned} M \ :&= \int_c^d \left|\frac{s-c}{d-c}+\frac{a-s}{b-a}\right|^q ds \\ &= \frac{1}{(b-a)(d-c)}\int_c^d |[(b-a)-(d-c)]\,s-cb+ad|^q\, ds \\ &= \frac{b-a-(d-c)}{(b-a)(d-c)}\int_c^d |s-s_0|^q\, ds \end{aligned}$$

since $b-a>d-c$ and

$$s_0 = \frac{cb-ad}{(b-a)-(d-c)} \in [c,d]. \tag{2.4}$$

Hence, as $c-s_0<0$ and $d-s_0>0$,

$$\begin{aligned} M &= \frac{b-a-(d-c)}{(b-a)(d-c)}\left[\int_c^{s_0}(s_0-s)^q\, ds + \int_{s_0}^d (s-s_0)^q\, ds\right] \\ &= \frac{b-a-(d-c)}{(b-a)(d-c)}\cdot\frac{(s_0-c)^{q+1}+(d-s_0)^{q+1}}{q+1}. \end{aligned}$$

Further simplification may be accomplished since

$$s_0-c = \frac{(c-a)(d-c)}{(b-a)(d-c)} \quad \text{and} \quad d-s_0 = \frac{(d-c)(b-d)}{(b-a)(d-c)},$$

giving

$$M = \frac{(d-c)^q}{(q+1)(b-a)[(b-a)-(d-c)]^q}\left[(c-a)^{q+1}+(b-d)^{q+1}\right].$$

Thus, combining the expression for M with (2.3) and using (2.2) gives (2.1) after some algebra.

Now, for the second inequality

$$|D(f;a,c,d,b)| \leq \sup_{s\in[a,b]} |K_{c,d}(s)| \int_a^b |f'(s)|\, ds.$$

From (1.3) it is easily seen that $K_{c,d}(s)$ is a piecewise linear and continuous function. It is negative on (a,s_0) and positive on (s_0,b). It reaches its extremities at c and d. Thus,

$$\begin{aligned} \sup_{s\in[a,b]} |K_{c,d}(s)| &= \max\left\{\frac{c-a}{b-a}, \frac{b-d}{b-a}\right\} \\ &= \frac{1}{b-a}\left\{\frac{c-a+b-d}{2} + \left|\frac{(c-a)-(b-d)}{2}\right|\right\}. \end{aligned}$$

A simple rearrangement gives the result as stated. ∎

Remark 1. *If we take $q = 1$ in (2.1) then the first inequality in (1.1) is recaptured.*

3. Some Inequalities for Mappings of Hölder Type

If we drop the assumption of absolute continuity and allow f to be Hölder continuous, then the following result is valid.

Theorem 4. *Assume that the mapping $f : [a,b] \to \mathbb{R}$ is of $r-H-$Hölder type, i.e.,*

$$|f(t) - f(s)| \leq H |t-s|^r \quad \text{for all } \; t, s \in [a,b], \tag{3.1}$$

where $r \in (0,1]$ and $H > 0$ are given.

Then for $a < c < d < b$, we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt \right| \leq \frac{(c-a)^{r+1} + (b-d)^{r+1}}{[(b-a)-(d-c)](r+1)} \cdot H. \tag{3.2}$$

The inequality (2.2) is best in the sense that we cannot put in the right hand side a constant K less than 1.

Proof. Write

$$\frac{1}{b-a} \int_a^b f(t)\,dt = \int_0^1 f(ub + (1-u)a)\,du$$

and similarly for the second term.

Then

$$\begin{aligned} I \;\; : &= \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt \\ &= \int_0^1 [f(ub + (1-u)a) - f(ud + (1-u)c)]\,du. \end{aligned}$$

Using the fact that f is of $r-H-$Hölder type, we have

$$\begin{aligned} |I| &\leq \int_0^1 |f(ub + (1-u)a) - f(ud + (1-u)c)|\,du \\ &\leq H \int_0^1 |ub + (1-u)a - ud - (1-u)c|^r\,du \\ &= H \int_0^1 |u[b-a-(d-c)] - (c-a)|^r\,du. \end{aligned} \tag{3.3}$$

Now, as $b-a>d-c$, then $u_0:=\frac{c-a}{(b-a)-(d-c)}\in(0,1)$ and

$$\begin{aligned}
&\int_0^1 \left|u\left[b-a-(d-c)\right]-(c-a)\right|^r du\\
=\;&\int_0^{u_0}\left[(c-a)-u\left[b-a-(d-c)\right]\right]^r du+\int_{u_0}^1\left[u\left[b-a-(d-c)\right]-(c-a)\right]^r du\\
=\;&-\frac{1}{b-a-(d-c)}\left.\frac{\left((c-a)-u\left[(b-a)-(d-c)\right]\right)^{r+1}}{r+1}\right|_0^{u_0}\\
&+\frac{1}{b-a-(d-c)}\left.\frac{\left(u\left[(b-a)-(d-c)\right]-(c-a)\right)^{r+1}}{r+1}\right|_{u_0}^1\\
=\;&-\frac{1}{(b-a)-(d-c)}\frac{\left((c-a)-u_0\left[(b-a)-(d-c)\right]\right)^{r+1}}{r+1}\\
&+\frac{(c-a)^{r+1}}{(r+1)\left(b-a-(d-c)\right)}+\frac{1}{(b-a)-(d-c)}\frac{\left(\left[b-a-(d-c)\right]-c+a\right)^{r+1}}{r+1}\\
&-\frac{\left(u_0\left[b-a-(d-c)\right]-(c-a)\right)^{r+1}}{\left[b-a-(d-c)\right](r+1)}\\
=\;&\frac{(c-a)^{r+1}+(b-d)^{r+1}}{\left[(b-a)-(d-c)\right](r+1)}.
\end{aligned}$$

Using (3.3) we deduce (3.2).

Assume that now, the inequality (2.2) holds with a constant $K>0$, i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(t)\,dt\right|\le K\cdot\frac{(c-a)^{r+1}+(b-d)^{r+1}}{\left[(b-a)-(d-c)\right](r+1)}H. \tag{3.4}$$

Choose $f_0:[0,1]\to\mathbb{R}$, $f_0(t)=t^r$, $r\in(0,1]$. Then

$$|f_0(t)-f_0(s)|\le|t-s|^r\quad\text{for all}\quad t,s\in[0,1],$$

which shows that f_0 is of $r-H-$Hölder type with $H=1$.

Now, choose in (3.4) $a=0$, $b=1$, $c\in(0,1)$, $d=c+\varepsilon$, ε small and such that $c+\varepsilon\in(0,1)$. Then we get

$$\left|\frac{1}{r+1}-\frac{(c+\varepsilon)^{r+1}-c^{r+1}}{\varepsilon(r+1)}\right|\le K\cdot\frac{c^{r+1}+(1-c-\varepsilon)^{r+1}}{(1-\varepsilon)(r+1)},$$

which is clearly equivalent with

$$\left|1-\frac{(c+\varepsilon)^{r+1}-c^{r+1}}{\varepsilon}\right|\le K\cdot\frac{c^{r+1}+(1-c-\varepsilon)^{r+1}}{1-\varepsilon}. \tag{3.5}$$

Now, if in (3.5) we let $\varepsilon\to0+$, then we get

$$|1-(r+1)c^r|\le K\left[c^{r+1}+(1-c)^{r+1}\right]\quad\text{for all}\quad c\in(0,1). \tag{3.6}$$

If in (3.6) we let $c\to0+$, then we get $1\le K$, and the theorem is completely proved. ∎

4. Results for the Riemann-Stieltjes Integral

The results obtained to date for bounds for differences of integral means assume that f is differentiable. That is, f is absolutely continuous. This assumption may be relaxed somewhat and bounds on $D(f;a,c,d,b)$ may still be procured. The following lemma holds.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be of bounded variation on $[a,b]$, then*

$$D(f;a,c,d,b)=\int_a^b K_{c,d}(s)\,df(s), \tag{4.1}$$

where $K_{c,d}$ is as given by (1.3) and $D(f;a,c,d,b)$ is as defined by (1.5).

Proof. The proof follows closely that used in obtaining (1.2). The integration by parts formula is used for Riemann-Stieltjes integrals to give

$$\begin{aligned}
& (b-a)(d-c)\int_a^b K_{c,d}(s)\,df(s) \\
= {} & (d-c)\int_a^c (a-s)\,df(s)+\int_c^d [(b-a)(s-c)-(d-c)(s-a)]\,df(s) \\
& +(d-c)\int_d^b (b-s)\,df(s) \\
= {} & (d-c)\left\{(a-s)f(s)\big]_a^c+\int_a^c f(s)\,ds\right\}+[(b-a)(s-c)-(d-c)(s-a)]f(s)\big]_c^d \\
& -[(b-a)-(d-c)]\int_c^d f(s)\,ds+(d-c)\left\{(b-s)f(s)\big]_d^b+\int_d^b f(s)\,ds\right\} \\
= {} & (d-c)\left\{(a-c)f(c)+\int_a^c f(s)\,ds\right\}+[(b-a)(d-c)-(d-c)(d-a)]f(d) \\
& +(d-c)(c-a)f(c)-[(b-a)-(d-c)]\int_c^d f(s)\,ds \\
& +(d-c)\left[\int_c^d f(s)\,ds-(b-d)f(d)\right] \\
= {} & (d-c)\int_c^d f(s)\,ds-(b-a)\int_c^d f(s)\,ds.
\end{aligned}$$

Division by $(b-a)(d-c)$ produces (4.1) on noting the definition (1.5). ∎

The following well known lemmas involving Riemann-Stieltjes integrals are well known. They are stated here for clarity. (See Cerone and Dragomir [4] where they were applied to three point rules in numerical integration.)

Lemma 2. *Let g, $v:[a,b]\to\mathbb{R}$ be such that g is continuous on $[a,b]$ and v is of bounded variation on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left|\int_a^b g(t)\,dv(t)\right|\le\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(v), \tag{4.2}$$

where $\bigvee_a^b(v)$ is the total variation of v on $[a,b]$.

Lemma 3. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is L-Lipschitzian on $[a,b]$. Then*

$$\left| \int_a^b g(t)\, dv(t) \right| \le L \int_a^b |g(t)|\, dt \tag{4.3}$$

with v being $L-$Lipschitzian if it satisfies

$$|v(x) - v(y)| \le L|x-y|$$

for all $x, y \in [a,b]$.

Lemma 4. *Let $g, v \in [a,b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then*

$$\left| \int_a^b g(t)\, dv(t) \right| \le \int_a^b |g(t)|\, dv(t). \tag{4.4}$$

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$. The following bounds hold*

$$\begin{aligned} &|D(f; a, c, d, b)| \\ &\le \begin{cases} \left[\dfrac{b-a-(d-c)}{2} + \left| \dfrac{c+d}{2} - \dfrac{a+b}{2} \right| \right] \dfrac{\bigvee_a^b(f)}{b-a}, & \\[2ex] \dfrac{(c-a)^2 + (b-d)^2}{2[(b-a)-(d-c)]} L, & \text{for } f\ L-\text{Lipschitzian;} \\[2ex] \left(\dfrac{b-d}{b-a} \right) f(b) - \left(\dfrac{c-a}{b-a} \right) f(a) & \\ \qquad + \left[\dfrac{c+d-(a+b)}{b-a} \right] f(s_0), & \text{for } f \text{ monotonic} \\ & \text{nondecreasing,} \end{cases} \end{aligned} \tag{4.5}$$

where $S_0 = \frac{cb-ad}{(b-a)-(d-c)}$.

Proof. Using Lemma 2, we have from (4.2)

$$\left| \int_a^b K_{c,d}(s)\, df(s) \right| \le \sup_{s\in[a,b]} |K_{c,d}(s)| \bigvee_a^b (f). \tag{4.6}$$

Now, $K_{c,d}(a) = K_{c,d}(b) = K_{c,d}(s_0) = 0$. Further, $K_{c,d}(s)$ consists of straight line segments on $[a,c]$, $[c,d]$ and $[d,b]$. The extreme values occur at c and d. Thus,

$$\begin{aligned} \sup_{s\in[a,b]} |K_{c,d}(s)| &= \max\{|K_{c,d}(c)|, |K_{c,d}(d)|\} = \max\left\{ \frac{c-a}{b-a}, \frac{b-d}{b-a} \right\} \\ &= \frac{1}{b-a} \left[\frac{c-a+b-d}{2} + \left| \frac{c-a-(b-d)}{2} \right| \right], \end{aligned}$$

which on rearrangement and using (4.6) gives the first inequality in (4.5).

Now, for the second inequality, we use Lemma 3 and so from (4.3)

$$\left| \int_a^b K_{c,d}(s)\, df(s) \right| \le L \int_a^b |K_{c,d}(s)|\, ds, \tag{4.7}$$

where

$$\int_a^b |K_{c,d}(s)|\,ds = \frac{1}{b-a}\int_a^c (s-a)\,ds + \frac{(b-a)-(d-c)}{(b-a)(d-c)}\int_c^d |s-s_0|\,ds + \frac{1}{b-a}\int_d^b (b-s)\,ds \tag{4.8}$$

with S_0 as given by (2.4).

Now,

$$\begin{aligned}\frac{1}{b-a}\int_a^c (s-a)\,ds &= \frac{(c-a)^2}{2(b-a)},\\ \frac{1}{b-a}\int_d^b (b-s)\,ds &= \frac{(b-d)^2}{2(b-a)}\end{aligned}$$

and

$$\begin{aligned}\int_c^d |s-s_0|\,ds &= \int_c^{s_0}(s_0-s)\,ds + \int_{s_0}^d (s-s_0)\,ds\\ &= \frac{(s_0-c)^2+(d-s_0)^2}{2}.\end{aligned}$$

That is, using (2.4), we have

$$\begin{aligned}&\frac{(b-a)-(d-c)}{(b-a)(d-c)}\int_c^d |s-s_0|\,ds\\ =\ &\frac{(d-c)}{2(b-a)[(b-a)-(d-c)]}\left[(c-a)^2+(b-d)^2\right],\end{aligned}$$

and so combining the above results and using (4.7) and (4.8) gives the second inequality.

For the final inequality in (4.5) we use Lemma 4 giving from (4.4), for f monotonic nondecreasing,

$$\left|\int_a^b K_{c,d}(s)\,df(s)\right| \le \int_a^b |K_{c,d}(s)|\,df(s). \tag{4.9}$$

Using the properties of $K_{c,d}(S)$ discussed earlier that $K_{c,d}(S)<0$ for $S\in(c,s_0)$ and $K_{c,d}(S)>0$ for $S\in(s_0,b)$ and zero at a, s_0 and b, then from (1.3)

$$\begin{aligned}&\int_a^b |K_{c,d}(s)|\,df(s)\\ =\ &\int_a^c \left(\frac{s-a}{b-a}\right)df(s) + \int_c^{s_0}\left(\frac{s-a}{b-a}+\frac{s-c}{d-c}\right)df(s)\\ &+\int_{s_0}^d \left(\frac{s-c}{d-c}+\frac{a-s}{b-a}\right)df(s) + \int_d^b\left(\frac{b-s}{b-a}\right)df(s).\end{aligned} \tag{4.10}$$

Now, integration by parts of the Riemann-Stieltjes integrals on the right of (4.10) produces, after some simplification,

$$\begin{aligned}\int_a^b |K_{c,d}(s)|\, df(s) &= -\frac{1}{b-a}\int_a^c f(s)\,ds + \left(\frac{1}{d-c}-\frac{1}{b-a}\right)\int_c^{s_0} f(s)\,ds \\ &\quad -\left(\frac{1}{d-c}-\frac{1}{b-a}\right)\int_{s_0}^d f(s)\,ds + \frac{1}{b-a}\int_d^b f(s)\,ds \\ &\le -\left(\frac{c-a}{b-a}\right) f(a) + \left(\frac{1}{d-c}-\frac{1}{b-a}\right)(s_0-c)\, f(s_0) \\ &\quad -\left(\frac{1}{d-c}-\frac{1}{b-a}\right)(d-s_0)\, f(s_0) + \left(\frac{b-d}{b-a}\right) f(b),\end{aligned}$$

where we have used the fact that f is monotonic nondecreasing to obtain the last inequality. Thus, from (3.5) we obtain the third inequality in (4.5) on grouping terms and simplifying. ∎

Remark 2. *From (4.10) we may use the fact that* $\sup_{s\in[a,b]} |K_{c,d}(S)|$ *occurs at* c *for* $s \in [a, s_0]$ *and at* d *for* $s \in [s_0, b]$ *to give for* f *monotonic nondecreasing:*

$$\begin{aligned}&\left|\int_a^b K_{c,d}(s)\, df(s)\right| \\ &\le \left(\frac{c-a}{b-a}\right)(f(c)-f(a)) + \left(\frac{c-a}{b-a}\right)(f(s_0)-f(c)) \\ &\quad + \left(1-\frac{d-a}{b-a}\right)(f(d)-f(s_0)) + \left(\frac{b-d}{b-a}\right)(f(b)-f(d)) \\ &= \left(\frac{c-a}{b-a}\right)(f(s_0)-f(a)) + \left(\frac{b-d}{b-a}\right)(f(b)-f(s_0)) \\ &= \left(\frac{b-d}{b-a}\right) f(b) - \left(\frac{c-a}{b-a}\right) f(a) + \frac{c+d-(a+b)}{b-a} f(s_0).\end{aligned}$$

Remark 3. *If we put* $r = 1$ *and* $H = L$ *in (4.2), then we obtain the second inequality in (4.5). It should also be noted that if the parallelogram identity*

$$2\left(x^2+y^2\right) = (x-y)^2 + (x+y)^2, \quad x, y \in \mathbb{R}$$

is used then

$$\frac{(c-a)^2+(b-d)^2}{2\left[(b-a)-(d-c)\right]} = \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2}-\frac{c+d}{2}}{(b-a)-(d-c)}\right)^2\right]\left[(b-a)-(d-c)\right],$$

with $\frac{1}{4}$ *being the best constant attained when* $\frac{a+b}{2} = \frac{c+d}{2}$.

Remark 4. *If we assume that there is a point* $x \in (a, b)$ *for which the functions is continuous, then we may recapture bounds for the Ostrowski functional*

$$\theta(f)(x) := f(x) - \mathcal{M}(f).$$

Indeed, if we assume that $c = x \in (a,b)$, $d = x + \varepsilon$ where $x + \varepsilon \in (a,b)$, then from (2.1) for example

$$\begin{aligned} & \left| D\left(f;a,x,x+\varepsilon ,b\right) \right| \\ = & \left| \mathcal{M}\left(f;a,b\right) -\frac{1}{\varepsilon }\int_{x}^{x+\varepsilon }f\left(u\right) du\right| \\ \leq & \frac{b-a}{\left(q+1\right) ^{\frac{1}{q}}}\left[1+\left(\frac{\varepsilon }{b-a-\varepsilon }\right) ^{q}\right] \left[\left(\frac{x-a}{b-a}\right) ^{q+1}+\left(\frac{b-x-\varepsilon }{b-a}\right) ^{q+1}\right] ^{\frac{1}{q}}\left\Vert f^{\prime }\right\Vert _{p}. \end{aligned}$$

Now, taking the limit as $\varepsilon \to 0+$ gives

$$\begin{aligned} \left| \theta \left(f\right) \left(x\right) \right| & \leq \frac{b-a}{\left(q+1\right) ^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right) ^{q+1}+\left(\frac{b-x}{b-a}\right) ^{q+1}\right] ^{\frac{1}{q}}\left\Vert f^{\prime }\right\Vert _{p}, \\ f^{\prime } & \in L_{p}\left[a,b\right] ,\ 1\leq p<\infty , \end{aligned}$$

recapturing a result of Dragomir and Wang [9]*.*

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References

[1] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. of Amer. Math. Soc.,* **123**(12) (1999), 3775-3781.

[2] N.S. BARNETT, P. CERONE, S.S. DRAGOMIR and A.M. FINK, Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_\infty [a,b]$ and applications, *Comp. and math. with Applics.* (accepted).

[3] P. CERONE and S.S. DRAGOMIR, On some inequalities arising from Montgomery's identity, *J. Comp. Anal. Applics.,* (accepted).

[4] P. CERONE and S.S. DRAGOMIR, Three point quadrature rules involving, at most, a first derivative, (submitted) (ON LINE: `http://rgmia.vu.edu.au/v2n4.html`).

[5] S.S. DRAGOMIR, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 59-66. (ON LINE: `http://rgmia.vu.edu.au/v2n1.html`).

[6] S.S. DRAGOMIR, Ostrowski inequality for monotonous mappings and applications, *J. KSIAM*, **3**(1) (1999), 127-135.

[7] S.S. DRAGOMIR, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.,* **38** (1999), 33-37.

[8] S.S. DRAGOMIR, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60** (1999), 495-508.

[9] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian Journal of Mathematics,* **40**(3) (1998), 299-304.

[10] A.M. FINK, Bounds on the derivative of a function from its averages, *Czech. Math. J.,* **42** (1992), 289-310.

[11] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht, 1994.

[12] A. OSTROWSKI, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

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