ON THE VALUE DISTRIBUTION OF $\varphi(z)f^{n-1}(z)f^{(k)}(z)$

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Abstract In this paper, the value distribution of $\varphi(z)f^{n-1}(z)f^{(k)}(z)$ is studied, where $f(z)$ is a transcendental meromorphic function, $\varphi(z)(\not\equiv 0)$ is a function such that $T(r,\varphi)=o(T(r,f))$ as $r\to+\infty$, n and k are positive integers such that $n=1$ or $n\geq k+3$. This generalizes a result of Hiong.

1. Introduction and the main result

In 1940, Milloux [5] showed that

Theorem A. *Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Further, let*

$$\phi(z)=\sum_{i=0}^{k}a_i(z)f^{(i)}(z),$$

where $a_i(z)(i=0,1,\ldots,k)$ are small functions of $f(z)$. Then we have

$$m\left(r,\frac{\phi}{f}\right)=S(r,f)$$

and

$$T(r,\phi)\leq(k+1)T(r,f)+S(r,f)$$

as $r\to+\infty$.

From this, it is easily for us to derive the following inequality which states a relationship between $T(r,f)$ and the 1-point of the derivatives of f. For the proof, please see [4], [7] or [8],

Theorem B. *Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Then*

$$\begin{aligned}T(r,f)\ &\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)\\&-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\end{aligned}$$

as $r\to+\infty$.

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In fact, the above estimate involves the consideration of the zeros and poles of $f(z)$. Then a natural question is: Is it possible to use only the counting functions of the zeros of $f(z)$ and an a-point of $f^{(k)}(z)$ to estimate the function $T(r,f)$? Hiong proved that the answer to this question is yes. Actually, Hiong [6] obtained the following inequality

Theorem C. *Let $f(z)$ be a non-constant meromorphic function. Further, let a, b and c be three finit complex numbers such that $b \neq 0$, $c \neq 0$ and $b \neq c$. Then*

$$\begin{aligned} T(r,f) &< N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f^{(k)}-b}\right) + N\left(r, \frac{1}{f^{(k)}-c}\right) \\ &\quad -N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \end{aligned}$$

as $r \to +\infty$.

Following this idea, a natural question to Theorem C is: Can we extend the three complex numbers to small functions of $f(z)$? In [9], by studying the zeros of the function $f(z)f'(z) - c(z)$, where $c(z)$ is a small function of $f(z)$, the author generalized the above inequality under an extra condition on the derivatives of $f^{(k)}(z)$. In fact, we have

Theorem D. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z)(\not\equiv 0)$ is a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$. Then for any finite non-zero distinct complex numbers b and c and any positive integer k such that $\varphi(z)f^{(k)}(z) \not\equiv$ constant, we have*

$$\begin{aligned} T(r,f) &< N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\varphi f^{(k)}-b}\right) + N\left(r, \frac{1}{\varphi f^{(k)}-c}\right) \\ &\quad -N(r,f) - N\left(r, \frac{1}{(\varphi f^{(k)})'}\right) + S(r,f) \end{aligned}$$

as $r \to +\infty$.

In this paper, we are going to show that Theorem D is still valid for all positive integers k. As a result, this generalizes Theorem C to small functions completely. More generally, we show that

Theorem. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z)(\not\equiv 0)$ is a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$. Suppose further that b and c are any finite non-zero distinct complex numbers, and k and n are positive integers. If $n = 1$ or $n \geq k+3$, then we have*

$$\begin{aligned} T(r,f) &< N\left(r, \frac{1}{f}\right) + \frac{1}{n}\left[N\left(r, \frac{1}{\varphi f^{n-1}f^{(k)}-b}\right) + N\left(r, \frac{1}{\varphi f^{n-1}f^{(k)}-c}\right)\right] \\ &\quad -\frac{1}{n}\left[N(r,f) + N\left(r, \frac{1}{(\varphi f^{n-1}f^{(k)})'}\right)\right] + S(r,f) \qquad (1) \end{aligned}$$

as $r \to +\infty$.

If $f(z)$ is entire, then (1) is true for all positive integers $n(\neq 2)$.

As an immedicate application of our theorem, we have

Corollary 1. *If we take $n = 1$ in the theorem, then we have Theorem D.*

Corollary 2. *If we take $n = 1$, $\varphi(z) \equiv 1$ and $f(z) = g(z) - a$, where a is any complex number, then we obtain Theorem C.*

Remark 1. We shall remark that our main theorem and corollaries are also valid if $f(z)$ is rational since $\varphi(z) \equiv$ *constant* and $\varphi(z)f^{n-1}(z)f^{(k)}(z) \not\equiv$ *constant* in this case.

Here, we assume that the readers are familiar with the basic concepts of the Nevanlinna value distribution theory and the notations $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$, $S(r, f)$ and etc., see e.g. [1].

2. Lemmae

For the proof of the main result, we need the following three lemmae.

Lemma 1. [3] *If $F(z)$ is a transcendental meromorphic function and $K > 1$, then there exists a set $M(K)$ of upper logarithmic density at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K - 1)) \exp(e(1 - K))\}$$

such that for every positive integer q,

$$\overline{\lim_{r\to\infty, r\notin M(K)}} \frac{T(r, F)}{T(r, F^{(q)})} \leq 3eK. \tag{2}$$

If $F(z)$ is entire, then we can replace $3eK$ by $2eK$ in (2).

Lemma 2. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z)(\not\equiv 0)$ is a meromorphic function such that $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$. Suppose further that k and n are positive integers. If $n = 1$ or $n \geq k + 3$, then $\varphi(z)f^{n-1}(z)f^{(k)}(z) \not\equiv$ constant.*

Proof: Without loss of generality, we suppose that the constant is 1. If $n = 1$, then $\varphi f^{(k)} \equiv 1$. Hence, $T(r, \varphi) = T(r, f^{(k)}) + O(1)$ as $r \to +\infty$ and this implies that

$$\overline{\lim_{r\to\infty, r\notin M(K)}} \frac{T(r, f)}{T(r, f^{(k)})} = \infty.$$

This contradicts Lemma (1).

If $n \geq k + 3$, then $T(r, \varphi f^{(k)}) = (n - 1)T(r, f)$ as $r \to +\infty$ and

$$(n - 1)T(r, f) \leq T(r, f^{(k)}) + S(r, f) \tag{3}$$

as $r \to +\infty$. On the other hand,

$$T(r, f^{(k)}) \le (k+1)T(r,f) + S(r,f) \tag{4}$$

as $r \to +\infty$. By (3) and (4), we have $n \le k+2$, a contradiction.

Hence, we have $\varphi f^{n-1} f^{(k)} \not\equiv constant$ in both cases and the lemma is proven.

Lemma 3. *If $f(z)$ is entire, then $\varphi(z)f^{n-1}(z)f^{(k)}(z) \not\equiv constant$ for all positive integers $n (\neq 2)$ and k.*

Proof: For the case $n = 1$, we still have $T(r,\varphi) = T(r, f^{(k)}) + O(1)$ as $r \to +\infty$, so a contradiction to Lemma (1) again.

For $n \ge 3$, instead of (4), we have

$$T(r, f^{(k)}) \le T(r,f) + S(r,f) \tag{5}$$

as $r \to +\infty$.

So by (3) and (5), we have $n \le 2$, a contradiction.

3. Proof of the main result

Proof: First of all, by the given conditions and Lemma 2, we know that $\varphi f^{n-1} f^{(k)} \not\equiv constant$ for $n \ge 1$. Therefore, we have

$$m\left(r, \frac{1}{\varphi f^n}\right) \le m\left(r, \frac{1}{\varphi f^{n-1} f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1). \tag{6}$$

From

$$m\left(r, \frac{1}{\varphi f^n}\right) = T(r, \varphi f^n) - N\left(r, \frac{1}{\varphi f^n}\right) + O(1),$$
$$m\left(r, \frac{1}{\varphi f^{n-1} f^{(k)}}\right) = T(r, \varphi f^{n-1} f^{(k)}) - N\left(r, \frac{1}{\varphi f^{n-1} f^{(k)}}\right) + O(1),$$

and (6), we have

$$\begin{aligned} T(r, \varphi f^n) \le{}& N\left(r, \frac{1}{\varphi f^n}\right) + T(r, \varphi f^{n-1} f^{(k)}) - N\left(r, \frac{1}{\varphi f^{n-1} f^{(k)}}\right) \\ &+ m\left(r, \frac{f^{(k)}}{f}\right) + O(1). \end{aligned} \tag{7}$$

Since $\varphi(z)f^{n-1}(z)f^{(k)} \not\equiv constant$, from the second fundamental theorem,

$$\begin{aligned} T(r, \varphi f^{n-1} f^{(k)}) <{}& N\left(r, \frac{1}{\varphi f^{n-1} f^{(k)}}\right) + N\left(r, \frac{1}{\varphi f^{n-1} f^{(k)} - b}\right) + \\ & N\left(r, \frac{1}{\varphi f^{n-1} f^{(k)} - c}\right) - N_1(r) + S(r, \varphi f^{(k)}) \end{aligned} \tag{8}$$

as $r \to +\infty$, where b and c are two non-zero distinct complex numbers and, as usual, $N_1(r)$ is defined as

$$N_1(r) = 2N(r, \varphi f^{n-1}f^{(k)}) - N(r, (\varphi f^{n-1}f^{(k)})') + N\left(r, \frac{1}{(\varphi f^{n-1}f^{(k)})'}\right).$$

Let z_0 be a pole of order $p \geq 1$ of f. Then $f^{n-1}f^{(k)}$ and $(f^{n-1}f^{(k)})'$ have a pole of order $k+np$ and $k+np+1$ at z_0 respectively. Thus $2(k+np)-(k+np+1) = k+np-1 \geq p$ and

$$N_1(r) \geq N(r, f) + N\left(r, \frac{1}{(\varphi f^{n-1}f^{(k)})'}\right) + S(r, f). \tag{9}$$

It is clear that $S(r, f^{(k)}) = S(r, f)$ and $m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. Thus by (7), (8) and (9),

$$\begin{aligned} T(r, \varphi f^n) &< N\left(r, \frac{1}{\varphi f^n}\right) + N\left(r, \frac{1}{\varphi f^{n-1}f^{(k)} - b}\right) + N\left(r, \frac{1}{\varphi f^{n-1}f^{(k)} - c}\right) \\ &\quad - N(r, f) - N\left(r, \frac{1}{(\varphi f^{n-1}f^{(k)})'}\right) + S(r, f) \end{aligned}$$

as $r \to +\infty$. Since $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$, we have the desired result.

If f is entire, then by Lemma (**??**), we still have $\varphi f^{n-1}f^{(k)} \not\equiv$ *constant* for all positive integers $n (\neq 2)$, (8) and (9). Thus the same arguement can be applied and the same result is obtained.

4. Concluding remarks and a conjecture

Remark 2. We expect that our theorem is also valid for the case $n = 2$ if $f(z)$ is entire.

Remark 3. In [10], Zhang studied the value distribution of $\varphi(z)f(z)f'(z)$ and he obtained the following result: *If $f(z)$ is a non-constant meromorphic function and $\varphi(z)$ is a non-zero meromorphic function such that $T(r, \varphi) = S(r, f)$ as $r \to +\infty$, then*

$$T(r, f) < \frac{9}{2}\overline{N}(r, f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi f f' - 1}\right) + S(r, f)$$

as $r \to +\infty$.

Hence, by this remark, we expect the following conjecture would be true.

Conjecture. *Let n and k be positive integers. If $n = 1$ or $n \geq k+3$, $f(z)$ is a non-constant meromorphic function and $\varphi(z)$ is a non-zero meromorphic function such that $T(r, \varphi) = S(r, f)$ as $r \to +\infty$, then*

$$T(r, f) < \frac{9}{2}\overline{N}(r, f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi f^{n-1}f^{(k)} - 1}\right) + S(r, f)$$

as $r \to +\infty$.

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