ON THE OSTROWSKI INEQUALITY FOR THE RIEMANN-STIELTJES INTEGRAL $\int_a^b f(t)\,du(t)$, WHERE f IS OF HÖLDER TYPE AND u IS OF BOUNDED VARIATION AND APPLICATIONS

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ABSTRACT. In this paper we point out an Ostrowski type inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is of $p-H-$Hölder type on $[a,b]$, and u is of bounded variation on $[a,b]$. Applications for the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also given.

1. Introduction

In 1938, A. Ostrowski proved the following integral inequality [1, p. 468]:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with its first derivative $f':(a,b)\to\mathbb{R}$ bounded on (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For a different proof than the original one provided by Ostrowski in 1938 as well as applications for special means (identric mean, logarithmic mean, $p-$logarithmic mean, etc.) and in *Numerical Analysis* for quadrature formulae of Riemann type, see the recent paper [2].

In [3], the following version of Ostrowski's inequality for the 1-norm of the first derivatives has been given.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with its first derivative $f':(a,b)\to\mathbb{R}$ integrable on (a,b), that is, $\|f'\|_1 := \int_a^b|f'(t)|\,dt<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_1, \tag{1.2}$$

for all $x\in[a,b]$.
The constant $\frac{1}{2}$ is sharp.

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Note that the sharpness of the constant $\frac{1}{2}$ in the class of differentiable mappings whose derivatives are integrable on (a,b) has been proven in the paper [5].

In [3], the authors applied (1.2) for special means and for quadrature formulae of Riemann type.

The following natural extension of Theorem 2 has been pointed out by S.S. Dragomir in [6].

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $\bigvee_a^b(f)$ its total variation on $[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\bigvee_a^b(f), \tag{1.3}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is sharp.

In [6], the author applied (1.3) for quadrature formulae of Riemann type as well as for Euler's Beta mapping.

In this paper we point out some generalizations of (1.3) for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation. Applications to the problem of approximating the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also given.

2. Some Integral Inequalities

The following theorem holds.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a $p-H-$Hölder type mapping, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^p,\ \text{for all } x,y\in[a,b]; \tag{2.1}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\begin{aligned}&\left|f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right| \\ &\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p\bigvee_a^b(u),\end{aligned} \tag{2.2}$$

for all $x\in[a,b]$, where $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{2}$ is the best possible, for all $p\in(0,1]$.

Proof. It is well known that if $g:[a,b]\to\mathbb{R}$ is continuous and $v:[a,b]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and the following inequality holds:

$$\left|\int_a^b g(t)\,dv(t)\right|\le\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(v). \tag{2.3}$$

Using this property, we have

$$\left|f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right| = \left|\int_a^b (f(x)-f(t))\,du(t)\right| \leq \sup_{t\in[a,b]}|f(x)-f(t)|\bigvee_a^b(u). \tag{2.4}$$

As f is of $p-H-$Hölder type, we have

$$\begin{aligned}\sup_{t\in[a,b]}|f(x)-g(t)| &\leq \sup_{t\in[a,b]}\left[H|x-t|^p\right] \\ &= H\max\left\{(x-a)^p,(b-x)^p\right\} \\ &= H\left[\max\left\{x-a,b-x\right\}\right]^p \\ &= H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p.\end{aligned}$$

Using (2.4), we deduce (2.2).
To prove the sharpness of the constant $\frac{1}{2}$ for any $p\in(0,1]$, assume that (2.2) holds with a constant $C>0$, that is,

$$\begin{aligned}&\left|f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right| \\ &\leq H\left[C(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p\bigvee_a^b(u),\end{aligned} \tag{2.5}$$

for all f, $p-H-$Hölder type mappings on $[a,b]$ and u of bounded variation on the same interval.
Choose $f(x)=x^p$ $(p\in(0,1])$, $x\in[0,1]$ and $u:[0,1]\to[0,\infty)$ given by

$$u(x)=\begin{cases}0 \text{ if } x\in[0,1) \\ 1 \text{ if } x=1\end{cases}.$$

As

$$|f(x)-f(y)|=|x^p-y^p|\leq|x-y|^p$$

for all $x,y\in[0,1]$, $p\in(0,1]$, it follows that f is of $p-H-$Hölder type with the constant $H=1$.
By using the integration by parts formula for Riemann-Stieltjes integrals, we have:

$$\begin{aligned}\int_0^1 f(t)\,du(t) &= f(t)\,u(t)]_0^1-\int_0^1 u(t)\,df(t) \\ &= 1-0=1\end{aligned}$$

and

$$\bigvee_0^1(u)=1.$$

Consequently, by (2.5), we get

$$|x^p-1|\leq\left[C+\left|x-\frac{1}{2}\right|\right]^p, \text{ for all } x\in[0,1].$$

For $x=0$, we get $1\leq\left(C+\frac{1}{2}\right)^p$, which implies that $C\geq\frac{1}{2}$, and the theorem is completely proved. ∎

The following corollaries are natural.

Corollary 1. *Let u be as in Theorem 4 and $f:[a,b]\to\mathbb{R}$ an $L-$Lipschitzian mapping on $[a,b]$, that is,*

$$|f(t)-f(s)|\leq L|t-s| \text{ for all } t,s\in[a,b] \tag{L}$$

where $L>0$ is fixed.
Then, for all $x\in[a,b]$, we have the inequality

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & L\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u) \end{aligned} \tag{2.6}$$

where

$$\Theta(f,u,x,a,b)=f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)$$

is the Ostrowski's functional associated to f and u as above. The constant $\frac{1}{2}$ is the best possible.

Remark 1. *If u is monotonic on $[a,b]$ and f is of $p-H-$Hölder type, then, by* (2.2) *we get*

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]|u(b)-u(a)|,\ x\in[a,b], \end{aligned} \tag{2.7}$$

and if we assume that f is $L-$Lipschitzian, then (2.6) *becomes*

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & L\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]|u(b)-u(a)|,\ x\in[a,b]. \end{aligned} \tag{2.8}$$

Remark 2. *If u is $K-$Lipschitzian, then obviously u is of bounded variation on $[a,b]$ and $\bigvee_a^b(u)\leq L(b-a)$. Consequently, if f is of $p-H-$Hölder type, then*

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & HK\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p(b-a),\ x\in[a,b] \end{aligned} \tag{2.9}$$

and if f is $L-$Lipschitzian, then

$$\begin{aligned} &|\Theta(f,u,a,b)| \\ \leq\ & LK\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](b-a),\ x\in[a,b]. \end{aligned} \tag{2.10}$$

The following corollary concerning a generalization of the mid-point inequality holds:

Corollary 2. *Let f and u be as defined in Theorem 4. Then we have the generalized mid-point formula*

$$|\Upsilon(f,u,a,b)|\leq\frac{H}{2^p}(b-a)^p\bigvee_a^b(u), \tag{2.11}$$

where

$$\Upsilon(f,u,a,b)=f\left(\frac{a+b}{2}\right)(u(b)-u(a))-\int_a^b f(t)\,du(t)$$

is the mid point functional associated to f *and* u *as above. In particular, if* f *is* $L-$*Lipschitzian, then*

$$|\Upsilon(f,u,a,b)|\leq\frac{L}{2}(b-a)\bigvee_a^b(u)\,. \tag{2.12}$$

Remark 3. *Now, if in* (2.11) *and* (2.12) *we assume that* u *is monotonic, then we get the midpoint inequalities*

$$|\Upsilon(f,u,a,b)|\leq\frac{H}{2^p}(b-a)^p|u(b)-u(a)| \tag{2.13}$$

and

$$|\Upsilon(f,u,a,b)|\leq\frac{L}{2}(b-a)|u(b)-u(a)| \tag{2.14}$$

respectively.
In addition, if in (2.11) *and* (2.12) *we assume that* u *is* $K-$*Lipschitzian, then we obtain the inequalities*

$$|\Upsilon(f,u,a,b)|\leq\frac{HK}{2^p}(b-a)^{p+1} \tag{2.15}$$

and

$$|\Upsilon(f,u,a,b)|\leq\frac{LK}{2}(b-a)^2\,. \tag{2.16}$$

The following inequalities of "*rectangle type*" also hold:

Corollary 3. *Let* f *and* u *be as in Theorem 4. Then we have the generalized "left rectangle" inequality*

$$\left|f(a)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\leq H(b-a)^p\bigvee_a^b(u) \tag{2.17}$$

and the "right rectangle" inequality

$$\left|f(b)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\leq H(b-a)^p\bigvee_a^b(u)\,. \tag{2.18}$$

Remark 4. *If we add* (2.17) *and* (2.18), *then, by the triangle inequality, we end up with the following generalized trapezoidal inequality*

$$\left|\frac{f(a)+f(b)}{2}(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\leq H(b-a)^p\bigvee_a^b(u)\,. \tag{2.19}$$

In what follows, we point out some results for the Riemann integral of a product.

Corollary 4. *Let* $f:[a,b]\to\mathbb{R}$ *be a* $p-H-$*Hölder type mapping and* $g:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned}&\left|f(x)\int_a^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\right| \\ \leq\;& H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p\int_a^b|g(s)|\,ds\end{aligned} \tag{2.20}$$

for all $x \in [a,b]$.

Proof. Define the mapping $u : [a,b] \to \mathbb{R}$, $u(t) = \int_a^t g(s)\,ds$. Then u is differentiable on (a,b) and $u'(t) = g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)\,g(t)\,dt$$

and

$$\bigvee_a^b (u) = \int_a^b |u'(t)|\,dt = \int_a^b |g(t)|\,dt.$$

Therefore, by the inequality (2.2), we deduce (2.20). ∎

Remark 5. *The best inequality we can get from* (2.20) *is that one for which* $x = \frac{a+b}{2}$, *obtaining the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt\right| \le \frac{1}{2^p} H (b-a)^p \int_a^b |g(s)|\,ds. \tag{2.21}$$

We now give some examples of weighted Ostrowski inequalities for some of the most popular weights.

Example 1. ***(Legendre)*** *If* $g(t) = 1$, *and* $t \in [a,b]$, *then we get the following Ostrowski inequality for Hölder type mappings* $f : [a,b] \to \mathbb{R}$

$$\left|(b-a) f(x) - \int_a^b f(t)\,dt\right| \le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^p (b-a) \tag{2.22}$$

for all $x \in [a,b]$, *and, in particular, the mid-point inequality*

$$\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \le \frac{1}{2^p} H (b-a)^{p+1}. \tag{2.23}$$

Example 2. ***(Logarithm)*** *If* $g(t) = \ln\left(\frac{1}{t}\right)$, $t \in (0,1]$, f *is of* $p-$*Hölder type on* $[0,1]$ *and the integral* $\int_0^1 f(t)\ln\left(\frac{1}{t}\right)dt$ *is finite, then we have*

$$\left|f(x) - \int_0^1 f(t)\ln\left(\frac{1}{t}\right)dt\right| \le H\left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]^p \tag{2.24}$$

for all $x \in [0,1]$ *and, in particular,*

$$\left|f\left(\frac{1}{2}\right) - \int_0^1 f(t)\ln\left(\frac{1}{t}\right)dt\right| \le \frac{1}{2^p} H. \tag{2.25}$$

Example 3. ***(Jacobi)*** *If* $g(t) = \frac{1}{\sqrt{t}}, t \in (0,1]$, f *is as above and the integral* $\int_0^1 \frac{f(t)}{\sqrt{t}}dt$ *is finite, then we have*

$$\left|f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt\right| \le H\left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]^p, \tag{2.26}$$

for all $x \in [0,1]$ *and, in particular,*

$$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt\right| \le \frac{1}{2^p} H. \tag{2.27}$$

Finally, we have the following:

Example 4. ***(Chebychev)*** *If* $g(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1,1)$, f *is of* $p-$*Hölder type on* $(-1,-1)$ *and the integral* $\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt$ *is finite, then*

$$\left| f(x) - \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq H \left[1 + |x|\right]^p \tag{2.28}$$

for all $x \in [-1,1]$, *and in particular,*

$$\left| f(0) - \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq H. \tag{2.29}$$

3. An Approximation for the Riemann-Stieltjes Integral

Consider $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ to be a division of the interval $[a,b]$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$. Define the general Riemann-Stieltjes sum

$$S(f, u, I_n, \xi) := \sum_{i=0}^{n-1} f(\xi_i)(u(x_{i+1}) - u(x_i)). \tag{3.1}$$

In what follows, we point out some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ by its Riemann-Stieltjes sum $S(f, u, I_n, \xi)$.

Theorem 5. *Let* $u : [a,b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *a* $p-H-$*Hölder type mapping. Then*

$$\int_a^b f(t)\, du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi), \tag{3.2}$$

where $S(f, u, I_n, \xi)$ *is as given in* (3.1) *and the remainder* $R(f, u, I_n, \xi)$ *satisfies the bound*

$$\begin{aligned} |R(f, u, I_n, \xi)| &\leq H\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^p \bigvee_a^b (u) \\ &\leq H[\nu(h)]^p \bigvee_a^b (u). \end{aligned} \tag{3.3}$$

Proof. We apply Theorem 4 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to obtain

$$\begin{aligned} &\left| f(\xi_i)(u(x_{i+1}) - u(x_i)) - \int_{x_i}^{x_{i+1}} f(t)\, du(t) \right| \\ &\leq H\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^p \bigvee_{x_i}^{x_{i+1}} (u), \end{aligned} \tag{3.4}$$

for all $i \in \{0, ..., n-1\}$.
Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we

deduce

$$\begin{aligned}
|R(f,u,I_n,\xi)| &\leq \sum_{i=0}^{n-1}\left|f(\xi_i)(u(x_{i+1})-u(x_i))-\int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \\
&\leq H\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p \bigvee_{x_i}^{x_{i+1}}(u) \\
&\leq H\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u).
\end{aligned}$$

However,

$$\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p \leq \left[\frac{1}{2}\nu(h)+\sup\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u)=\bigvee_a^b(u),$$

which completely proves the first inequality in (3.3).
For the second inequality, we observe that

$$\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}\cdot h_i,$$

for all $i\in\{0,...,n-1\}$.
The theorem is thus proved. ∎

The following corollaries are natural.

Corollary 5. *Let u be as in Theorem 5 and f an $L-$Lipschitzian mapping. Then we have the formula* (3.2) *and the remainder $R(f,u,I_n,\xi)$ satisfies the bound*

$$\begin{aligned}
|R(f,u,I_n,\xi)| &\leq L\left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b(u) \\
&\leq H\nu(h)\bigvee_a^b(u).
\end{aligned}\tag{3.5}$$

Remark 6. *If u is monotonic on $[a,b]$, then the error estimate* (3.3) *becomes*

$$\begin{aligned}
&|R(f,u,I_n,\xi)| \\
&\leq H\left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^p|u(b)-u(a)| \\
&\leq H[\nu(h)]^p|u(b)-u(a)|
\end{aligned}\tag{3.6}$$

and (3.5) *becomes*

$$\begin{aligned}
&|R(f,u,I_n,\xi)| \\
&\leq L\left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]|u(b)-u(a)| \\
&\leq L\nu(h)|u(b)-u(a)|.
\end{aligned}\tag{3.7}$$

Using Remark 2, we can state the following corollary.

Corollary 6. *If $u : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant K and $f : [a,b] \to \mathbb{R}$ is of $p-H-$Hölder type, then the formula* (3.2) *holds and the remainder $R(f,u,I_n,\xi)$ satisfies the bound*

$$\begin{aligned} (3.8) \qquad |R(f,u,I_n,\xi)| &\leq HK \sum_{i=0}^{n-1} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^p h_i \\ &\leq HK \sum_{i=0}^{n-1} h_i^{p+1} \leq HK (b-a) [\nu(h)]^p . \end{aligned}$$

In particular, if we assume that f is $L-$Lipschitzian, then

$$\begin{aligned} (3.9) \qquad |R(f,u,I_n,\xi)| &\leq \frac{1}{2} LK \sum_{i=0}^{n-1} h_i^2 + LK \sum_{i=0}^{n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| h_i \\ &\leq LK \sum_{i=0}^{n-1} h_i^2 \leq LK (b-a) \nu(h) . \end{aligned}$$

The best quadrature formula we can get from Theorem 5 is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ for all $i \in \{0, ..., n-1\}$. Consequently, we can state the following corollary.

Corollary 7. *Let f and u be as in Theorem 5. Then*

$$(3.10) \qquad \int_a^b f(t)\, du(t) = S_M(f,u,I_n) + R_M(f,u,I_n)$$

where $S_M(f,u,I_n)$ is the generalized midpoint formula, that is;

$$S_M(f,u,I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2} \right) (u(x_{i+1}) - u(x_i))$$

and the remainder satisfies the estimate

$$(3.11) \qquad |R_M(f,u,I_n)| \leq \frac{H}{2^p} [\nu(h)]^p \bigvee_a^b (u) .$$

In particular, if f is $L-$Lipschitzian, then we have the bound:

$$(3.12) \qquad |R_M(f,u,I_n)| \leq \frac{H}{2} \nu(h) \bigvee_a^b (u) .$$

Remark 7. *If in* (3.11) *and* (3.12) *we assume that u is monotonic, then we get the inequalities*

$$(3.13) \qquad |R_M(f,u,I_n)| \leq \frac{H}{2^p} [\nu(h)]^p |f(b) - f(a)|$$

and

$$(3.14) \qquad |R_M(f,u,I_n)| \leq \frac{H}{2} \nu(h) |f(b) - f(a)| .$$

The case where f is $K-$Lipschitzian is embodied in the following corollary.

Corollary 8. *Let u and f be as in Corollary 6. Then we have the quadrature formula* (3.10) *and the remainder satisfies the estimate*

$$|R_M(f,u,I_n)| \leq \frac{HK}{2^p}\sum_{i=0}^{n-1} h_i^{p+1} \leq \frac{HK}{2^p}[\nu(h)]^p. \tag{3.15}$$

In particular, if f is $L-$Lipschitzian, then we have the estimate

$$|R_M(f,u,I_n)| \leq \frac{1}{2}LK\sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{2}LK(b-a)\nu(h). \tag{3.16}$$

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