A PERTURBED TRAPEZOID INEQUALITY IN TERMS OF THE THIRD DERIVATIVE AND APPLICATIONS

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Abstract. Some error estimates in terms of the $p-$norms of the third derivative for the remainder in a perturbed trapezoid formula are given. Applications to composite quadrature formulae, for the expectation of a random variable and for Hermite-Hadamard divergence in Information Theory are pointed out.

1. Introduction

In [4], by the use of Grüss' integral inequality, the authors have obtained the following perturbed trapezoid inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) and asume that*

$$\gamma := \inf_{x\in(a,b)} f''(x) > -\infty \text{ and } \Gamma := \sup_{x\in(a,b)} f''(x) < \infty.$$

Then we have the inequality

$$\left| \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \leq \frac{(b-a)^3}{12}(\Gamma-\gamma). \tag{1.1}$$

Using a finer argument based on a pre-Grüss inequality, Cerone and Dragomir [2, p. 121] improved the above result as follows.

Theorem 2. *Let f have the properties of Theorem 1. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \leq \frac{1}{24\sqrt{5}}(b-a)^3(\Gamma-\gamma). \tag{1.2}$$

The main aim of the present work is to obtain some bounds for the left part of (1.2) in terms of the $p-$norms of f''' assuming that the function f is twice differentiable on (a,b) and that the second derivative is absolutely continuous on (a,b).

A number of applications are also pointed out.

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2. A Perturbed Trapezoid Formula

The following representation lemma holds.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be such that the second derivative is absolutely continuous on $[a,b]$. Then we have the equality:*

$$\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right] \tag{2.1}$$
$$= \frac{1}{4(b-a)}\int_a^b\int_a^b\left(\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right)(t-s)\,dtds.$$

Proof. Integrating by parts, we have

$$\begin{aligned}
I &:= \int_a^b\int_a^b\left(\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right)(t-s)\,dtds \\
&= \int_a^b\int_a^b\left[\frac{f''(t)+f''(s)}{2}(t-s)-\int_s^t f''(u)\,du\right](t-s)\,dtds \\
&= \int_a^b\int_a^b\left[\frac{f''(t)(t-s)^2+f''(s)(t-s)^2}{2}-(f'(t)-f'(s))(t-s)\right]dtds \\
&= \frac{1}{2}\left[\int_a^b\int_a^b f''(t)(t-s)^2\,dtds+\int_a^b\int_a^b f''(s)(t-s)^2\,dtds\right] \\
&\quad -\int_a^b\int_a^b (f'(t)-f'(s))(t-s)\,dtds.
\end{aligned}$$

By symmetry,

$$J := \int_a^b\int_a^b f''(t)(t-s)^2\,dtds = \int_a^b\int_a^b f''(s)(t-s)^2\,dtds,$$

and using Korkine's identity or direct computation, we have

$$\begin{aligned}
K &:= \int_a^b\int_a^b (f'(t)-f'(s))(t-s)\,dtds \\
&= 2\left[(b-a)\int_a^b f'(t)\,tdt-\int_a^b f'(t)\,dt\cdot\int_a^b tdt\right].
\end{aligned}$$

Then, $I = J - K$.

Since

$$\begin{aligned}
J &= \int_a^b f''(t)\left(\int_a^b (t-s)^2\,ds\right)dt \\
&= \frac{1}{3}\left[\int_a^b f''(t)(b-t)^3\,dt+\int_a^b (t-a)^3 f''(t)\,dt\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\left[f'(t)(b-t)^3\Big|_a^b + 3\int_a^b (b-t)^2 f'(t)\,dt \right. \\
&\quad \left. + f'(t)(t-a)^3\Big|_a^b - 3\int_a^b (t-a)^2 f'(t)\,dt \right] \\
&= \frac{1}{3}\left[-f'(a)(b-a)^3 + 3\left[f(t)\,(b-t)^2\Big|_a^b - 2\int_a^b (b-t) f(t)\,dt \right] \right. \\
&\quad \left. + f'(b)(b-a)^3 - 3\left[f(t)\,(t-a)^2\Big|_a^b - 2\int_a^b (t-a) f(t)\,dt \right] \right] \\
&= \frac{1}{3}\left[[f'(b)-f'(a)](b-a)^3 + 3\left[-f(a)(b-a)^2 + 2\int_a^b (b-t) f(t)\,dt \right] \right. \\
&\quad \left. - 3\left[f(b)(b-a)^2 - 2\int_a^b (t-a) f(t)\,dt \right] \right] \\
&= \frac{1}{3}[f'(b)-f'(a)](b-a)^3 - \left[\frac{f(a)+f(b)}{2} \right](b-a)^2 + 2(b-a)\int_a^b f(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
K &= 2\left[(b-a)\left[f(t)\,t\Big|_a^b - \int_a^b f(t)\,dt \right] - [f(b)-f(a)]\frac{b^2-a^2}{2} \right] \\
&= 2\left[(b-a)\left[f(b)\,b - f(a)\,a - \int_a^b f(t)\,dt \right] - (b-a)[f(b)-f(a)]\frac{a+b}{2} \right] \\
&= 2(b-a)\left[f(b)\,b - f(a)\,a - \int_a^b f(t)\,dt - f(b)\frac{a+b}{2} + f(a)\frac{a+b}{2} \right] \\
&= (b-a)^2[f(a)+f(b)] - 2(b-a)\int_a^b f(t)\,dt,
\end{aligned}
$$

then

$$
I = \frac{1}{3}[f'(b)-f'(a)](b-a)^3 - 2[f(a)+f(b)](b-a)^2 + 4(b-a)\int_a^b f(t)\,dt
$$

Dividing by $4(b-a)$ we deduce the desired equality (2.1). ∎

The following perturbed version of the trapezoid inequality holds.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be such that the second derivative is absolutely continuous on $[a,b]$. Then we have*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)+\frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \tag{2.2}$$

$$\leq\begin{cases}\frac{1}{16(b-a)}\int_a^b\int_a^b|t-s|^3\left\|f'''\right\|_{[t,s],\infty}dtds & \text{if } f'''\in L_\infty[a,b];\\ \frac{1}{8(q+1)^{\frac{1}{q}}(b-a)}\int_a^b\int_a^b|t-s|^{2+\frac{1}{q}}\left\|f'''\right\|_{[t,s],p}dtds & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{8(b-a)}\int_a^b\int_a^b(t-s)^2\left\|f'''\right\|_{[t,s],1}dtds\end{cases}$$

$$\leq\begin{cases}\frac{(b-a)^4}{160}\left\|f'''\right\|_{[a,b],\infty} & \text{if } f'''\in L_\infty[a,b];\\ \frac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\left\|f'''\right\|_{[a,b],p} & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{(b-a)^3}{48}\left\|f'''\right\|_{[a,b],1},\end{cases}$$

where $\|h\|_{[c,d],l}:=\left|\int_c^d|h(u)|^l du\right|^{\frac{1}{l}}$ *if* $l\geq1$ *and* $\|h\|_{[c,d],\infty}=ess\sup_{\substack{u\in[c,d]\\(u\in[d,c])}}|h(u)|$.

Proof. Denote

$$R(f;a,b):=\frac{1}{4(b-a)}\int_a^b\int_a^b\left(\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right)(t-s)\,dtds.$$

As

$$\begin{aligned}\left|\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right| &\leq \left\|f'''\right\|_{[t,s],\infty}\left|\int_s^t\left|u-\frac{t+s}{2}\right|du\right|\\ &= \frac{(t-s)^2}{4}\left\|f'''\right\|_{[t,s],\infty},\end{aligned}$$

$$\begin{aligned}&\left|\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right|\\ &\qquad\leq\left\|f'''\right\|_{[t,s],p}\left|\int_s^t\left|u-\frac{t+s}{2}\right|^q du\right|^{\frac{1}{q}}\\ &\qquad=\frac{|t-s|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'''\right\|_{[t,s],q},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\end{aligned}$$

and

$$\left|\int_s^t\left(u-\frac{t+s}{2}\right)f'''(u)\,du\right|\leq\sup_{\substack{u\in[t,s]\\(u\in[t,s])}}\left|u-\frac{t+s}{2}\right|\left\|f'''\right\|_{[t,s],1}$$

then we can state that

$$\left|\int_s^t \left(u-\frac{t+s}{2}\right) f'''(u)\,du\right| \tag{2.3}$$

$$\le \begin{cases} \frac{(t-s)^2}{4}\left\|f'''\right\|_{[t,s],\infty} & \text{if } f'''\in L_\infty[a,b]; \\ \frac{|t-s|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left\|f'''\right\|_{[t,s],q} & \text{if } f'''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|t-s|}{2}\left\|f'''\right\|_{[t,s],1}. \end{cases}$$

Taking the modulus of $R(f;a,b)$ we get, by (2.3),

$$\begin{aligned} &|R(f;a,b)| \\ \le\ & \frac{1}{4(b-a)}\int_a^b\int_a^b |t-s|\left|\int_s^t \left(u-\frac{t+s}{2}\right)f'''(u)\,du\right| dtds \\ \le\ & \frac{1}{4(b-a)}\begin{cases} \frac{1}{4}\int_a^b\int_a^b |t-s|^3\left\|f'''\right\|_{[t,s],\infty} dtds & \text{if } f'''\in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}}\int_a^b\int_a^b |t-s|^{2+\frac{1}{q}}\left\|f'''\right\|_{[t,s],p} dtds & \text{if } f'''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\int_a^b\int_a^b |t-s|^2\left\|f'''\right\|_{[t,s],1} dtds \end{cases} \end{aligned}$$

which proves the first inequality in (2.2).

Now, consider the double integral

$$\begin{aligned} I_m &:= \int_a^b\int_a^b |t-s|^m\,dtds = \int_a^b\left[\int_a^t (t-s)^m\,ds + \int_t^b (s-t)^m\,ds\right]dt \\ &= \int_a^b \left[\frac{(t-a)^{m+1}+(b-t)^{m+1}}{m+1}\right]dt = \frac{2(b-a)^{m+2}}{(m+1)(m+2)} \end{aligned}$$

for all $m>0$.

Using the above calculation for I_m, we have:-

$$\begin{aligned} \int_a^b\int_a^b |t-s|^3\left\|f'''\right\|_{[t,s],\infty} dtds &\le \left\|f'''\right\|_{[a,b],\infty}\int_a^b\int_a^b |t-s|^3\,dtds \\ &= \frac{(b-a)^5}{10}\cdot\left\|f'''\right\|_{[a,b],\infty}, \\ \int_a^b\int_a^b |t-s|^{2+\frac{1}{q}}\left\|f'''\right\|_{[t,s],p} dtds &\le \left\|f'''\right\|_{[a,b],p}\int_a^b\int_a^b |t-s|^{2+\frac{1}{q}}\,dtds \\ &= \frac{2q^2(b-a)^{4+\frac{1}{q}}}{(3q+1)(4q+1)}\cdot\left\|f'''\right\|_{[a,b],p} \end{aligned}$$

and

$$\begin{aligned} \int_a^b\int_a^b (t-s)^2\left\|f'''\right\|_{[t,s],1} dtds &\le \left\|f'''\right\|_{[a,b],1}\int_a^b\int_a^b (t-s)^2\,dtds \\ &= \frac{(b-a)^4}{6}\cdot\left\|f'''\right\|_{[a,b],1}, \end{aligned}$$

which give the last part of (2.2). ∎

3. Applications to Composite Quadrature Formulae

Consider the division $I_n : a = x_0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n = b$ and define $h_i := x_{i+1} - x_i$ $\left(i = \overline{0, n-1}\right)$ and $\nu(I_n) := \max\left\{h_i | i = \overline{0, n-1}\right\}$. If

$$P_n(f; I_n) := \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]h_i - \frac{1}{12}\sum_{i=0}^{n-1} h_i^2 [f'(x_{i+1}) - f'(x_i)] \tag{3.1}$$

is the perturbed trapezoid formula associated with the absolutely continuous function $f : [a, b] \to \mathbb{R}$, then we may state the following theorem.

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be such that the second derivative is absolutely continuous on $[a, b]$. Then, for a given division I_n, we have:*

$$\int_a^b f(t)\,dt = P_n(f; I_n) + R_n(f; I_n), \tag{3.2}$$

where $P_n(f; I_n)$ is given in (3.1) and the remainder $R_n(f; I_n)$ satisfies the estimate:

$$|R_n(f; I_n)| \le \begin{cases} \frac{1}{160}\|f'''\|_{[a,b],\infty}\sum_{i=0}^{n-1} h_i^4 \quad \text{if} \quad f''' \in L_\infty[a, b]; \\ \\ \frac{q^2}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\|f'''\|_{[a,b],p}\left(\sum_{i=0}^{n-1} h_i^{3q+1}\right)^{\frac{1}{q}} \\ \qquad \text{if} \quad f''' \in L_p[a, b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{48}[\nu(h)]^3\|f'''\|_{[a,b],1}. \end{cases} \tag{3.3}$$

Proof. If we apply Theorem 3 to the intervals $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$ we get

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{f(x_i) + f(x_{i+1})}{2}\cdot h_i + \frac{h_i^2}{12}[f'(x_{i+1}) - f'(x_i)]\right|$$
$$\le \begin{cases} \dfrac{h_i^4}{160}\|f'''\|_{[x_i,x_{i+1}],\infty} & \text{if} \quad f''' \in L_\infty[a, b]; \\ \\ \dfrac{q^2 h_i^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\|f'''\|_{[x_i,x_{i+1}],p} & \text{if} \quad f''' \in L_p[a, b], \\ & \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \dfrac{h_i^3}{48}\|f'''\|_{[x_i,x_{i+1}],1}. \end{cases}$$

Using the generalised triangle inequality, we obtain

$$(3.4)\qquad \left|R_n\left(f;I_n\right)\right|$$

$$= \left|\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} f\left(t\right)dt - \frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2}\cdot h_i + \frac{h_i^2}{12}\left[f'\left(x_{i+1}\right)-f'\left(x_i\right)\right]\right]\right|$$

$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f\left(t\right)dt - \frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2}\cdot h_i + \frac{h_i^2}{12}\left[f'\left(x_{i+1}\right)-f'\left(x_i\right)\right]\right|$$

$$\leq \begin{cases} \frac{1}{160}\sum_{i=0}^{n-1} h_i^4\left\|f'''\right\|_{[x_i,x_{i+1}],\infty} & \text{if } \quad f'''\in L_\infty\left[a,b\right]; \\ \frac{q^2}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1} h_i^{3+\frac{1}{q}}\left\|f'''\right\|_{[x_i,x_{i+1}],p} & \text{if } \quad f'''\in L_p\left[a,b\right], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{48}\sum_{i=0}^{n-1} h_i^3\left\|f'''\right\|_{[x_i,x_{i+1}],1}. \end{cases}$$

As

$$\sum_{i=0}^{n-1} h_i^4\left\|f'''\right\|_{[x_i,x_{i+1}],\infty} \leq \left\|f'''\right\|_{[a,b],\infty}\sum_{i=0}^{n-1} h_i^4,$$

then by (3.4) we deduce the first inequality in (3.3).

Using Hölder's discrete inequality, we may write

$$\sum_{i=0}^{n-1} h_i^{3+\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}\left|f'''\left(t\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=0}^{n-1} h_i^{\left(3+\frac{1}{q}\right)q}\right)^{\frac{1}{q}}\times\left(\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}}\left|f'''\left(t\right)\right|^p dt\right)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=0}^{n-1} h_i^{3q+1}\right)^{\frac{1}{q}}\times\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|f'''\left(t\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=0}^{n-1} h_i^{3q+1}\right)^{\frac{1}{q}}\left\|f'''\right\|_{[a,b],p},$$

which proves the second inequality in (3.3).

Finally, we observe that

$$\sum_{i=0}^{n-1} h_i^3\left\|f'''\right\|_{[x_i,x_{i+1}],1} \leq \left[\nu\left(h\right)\right]^3\sum_{i=0}^{n-1}\left\|f'''\right\|_{[x_i,x_{i+1}],1} = \left[\nu\left(h\right)\right]^3\left\|f'''\right\|_{[a,b],1},$$

and the theorem is proved. ∎

In practical applications, it is useful to consider an equidistant partitioning

$$E_n : x_i := a + i\cdot\frac{b-a}{n},\quad i=0,\ldots,n.$$

In this case the perturbed trapezoid formula becomes:

$$\begin{aligned}(3.5)\quad P_n(f) &:= \frac{b-a}{2n}\sum_{i=0}^{n-1}\left[f\left(a+i\cdot\frac{b-a}{n}\right)+f\left(a+(i+1)\cdot\frac{b-a}{n}\right)\right] \\ &\quad -\frac{(b-a)^2}{12n^2}\left[f'(b)-f'(a)\right].\end{aligned}$$

Consequently, the following corollary holds.

Corollary 1. *If f is as in Theorem 4, then we have*

$$\int_a^b f(t)\,dt = P_n(f)+R_n(f), \tag{3.6}$$

where $P_n(f)$ is given in (3.5) and the remainder $R_n(f)$ satisfies the estimate

$$\begin{aligned}(3.7)\qquad & |R_n(f)| \\ & \leq \begin{cases} \dfrac{(b-a)^4}{160n^3}\|f'''\|_{[a,b],\infty} & \text{if } f'''\in L_\infty[a,b]; \\ \dfrac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}n^3}\|f'''\|_{[a,b],p} & \\ \qquad \text{if } f'''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \dfrac{(b-a)^3}{48n^3}\|f'''\|_{[a,b],1}. & \end{cases}\end{aligned}$$

Remark 1. *It is important to note that the perturbed trapezoid formula contains, in addition to the classical trapezoid formula, the term $-\frac{(b-a)^2}{12n^2}[f'(b)-f'(a)]$, which can be calculated simply when the derivatives of the end-points a and b are known. As can be seen in formula (3.7), the order of the new formula is 3, while the order of the classical trapezoid formula is only 2.*

Remark 2. *Atkinson* [1] *terms the quadrature rule (2.2) a connected trapezoidal rule and obtains it using an asymptotic error estimate approach which does not provide an expression for the error bound. He does, however, state that the corrected trapezoidal rule is $O(h^4)$ compared with $O(h^2)$ for the trapezoidal rule.*

4. Applications for Expectation

Let X be a random variable having the p.d.f., $f:[a,b]\to\mathbb{R}$ and the *cumulative distribution function* $F:[a,b]\to[0,1]$, i.e.,

$$F(x)=\int_a^x f(t)\,dt,\quad x\in[a,b].$$

Theorem 5. *With the above assumptions and if the p.d.f., f is differentiable on $[a,b]$ and f' is absolutely continuous, then*

$$\left|E(X)-\frac{a+b}{2}-\frac{(b-a)^2}{12}\left[f(b)-f(a)\right]\right| \tag{4.1}$$

$$\leq \begin{cases} \frac{(b-a)^4}{160}\left\|f''\right\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\left\|f''\right\|_{[a,b],p} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^3}{48}\left\|f''\right\|_{[a,b],1}, \end{cases}$$

where $E(X)$ is the expectation of X.

Proof. Applying Theorem 3 for F, we may write that

$$\left|\int_a^b F(t)\,dt-\frac{F(a)+F(b)}{2}(b-a)+\frac{(b-a)^2}{12}\left[f(b)-f(a)\right]\right| \tag{4.2}$$

$$\leq \begin{cases} \frac{(b-a)^4}{160}\left\|f''\right\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{q^2(b-a)^{3+\frac{1}{q}}}{4(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}\left\|f''\right\|_{[a,b],p} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^3}{48}\left\|f''\right\|_{[a,b],1}, \end{cases}$$

However, $F(a)=0$, $F(b)=1$ and

$$\int_a^b F(t)\,dt=b-E(X),$$

and then by (4.2) we obtain the desired inequality (4.1). ∎

5. Applications for Hermite-Hadamard Divergence

Assume that a set χ and the $\sigma-$finite measure μ is given. Consider the set of all probability densities on μ to be $\Omega:=\left\{p|p:\chi\to\mathbb{R},\, p(x)\geq 0,\ \int_\chi p(x)\,d\mu(x)=1\right\}$.

Csiszár $f-$divergence is defined as follows [3]

$$D_f(p,q):=\int_\chi p(x)\,f\left[\frac{q(x)}{p(x)}\right]d\mu(x),\quad p,q\in\Omega, \tag{5.1}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$.

By appropriately defining this convex function, various divergences such as: Kullback-Leibler divergence, variation distance D_v, Hellinger discrimination D_H, χ^2-divergence D_{χ^2}, $\alpha-$divergence D_α, Bhattacharyya distance D_B, Harmonic distance D_{Ha}, Jeffreys distance D_J, triangular discrimination D_Δ, etc. (see [5]).

In [6], Shioya and Da-te introduced the generalised Lin-Wong $f-$divergence $D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right)$ and the Hermite-Hadamard (HH) divergence

$$D_{HH}^{f}(p,q):=\int_{\chi}p(x)\frac{\int_{1}^{\frac{q(x)}{p(x)}}f(t)\,dt}{\frac{q(x)}{p(x)}-1}d\mu(x),\quad p,q\in\Omega, \tag{5.2}$$

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

$$D_f\left(p,\frac{1}{2}p+\frac{1}{2}q\right)\leq D_{HH}^{f}(p,q)\leq\frac{1}{2}D_f(p,q), \tag{5.3}$$

provided that f is convex and normalised, i.e., $f(1)=0$.

The following result holds.

Theorem 6. *Let $0\leq r\leq 1\leq R<\infty$ and $f:[r,R]\to\mathbb{R}$ be a twice differentiable function so that the second derivative $f'':[r,R]\to\mathbb{R}$ is absolutely continuous on $[r,R]$. If $p,q\in\Omega$ and $r\leq\frac{q(x)}{p(x)}\leq R$ for a.e. $x\in\chi$, then we have the inequality:*

$$\left|D_{HH}^{f}(p,q)-\frac{1}{2}D_f(p,q)+\frac{1}{12}D_{(\cdot-1)f'(\cdot)}(p,q)\right| \tag{5.4}$$

$$\leq\begin{cases}\frac{1}{160}\int_{\chi}\frac{|q(x)-p(x)|^{3}}{p^{2}(x)}\|f'''\|_{\left[\frac{q(x)}{p(x)},1\right],\infty}d\mu(x)\\[2ex] \frac{\beta^{2}}{4(3\beta+1)(4\beta+1)(\beta+1)^{\frac{1}{\beta}}}\int_{\chi}\frac{|q(x)-p(x)|^{2+\frac{1}{\beta}}}{(p(x))^{1+\frac{1}{\beta}}}\|f'''\|_{\left[\frac{q(x)}{p(x)},1\right],\alpha}d\mu(x)\\ \qquad\qquad \textit{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\ f'''\in L_{\alpha}[r,R];\\[2ex] \frac{1}{48}\int_{\chi}\frac{(q(x)-p(x))^{2}}{p(x)}\|f'''\|_{\left[\frac{q(x)}{p(x)},1\right],1}d\mu(x)\end{cases}$$

$$\leq\begin{cases}\frac{\|f'''\|_{[r,R],\infty}}{160}D_{|\chi|^{3}}(p,q) & \textit{if}\quad f'''\in L_{\infty}[r,R];\\[2ex] \frac{\beta^{2}\|f'''\|_{[r,R],\alpha}}{4(3\beta+1)(4\beta+1)(\beta+1)^{\frac{1}{\beta}}}D_{|\chi|^{2+\frac{1}{\beta}}}(p,q) & \textit{if}\quad f'''\in L_{\alpha}[r,R],\\ & \qquad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \frac{\|f'''\|_{[r,R],1}}{48}D_{\chi^{2}}(p,q),\end{cases}$$

where

$$D_{|\chi|^{m}}(p,q):=\int_{\chi}\frac{|q(x)-p(x)|^{m}}{[p(x)]^{m-1}}d\mu(x),\quad m\in\mathbb{R},\ m>0.$$

Proof. We use the inequality (2.2) in the following version:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}+\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \tag{5.5}$$
$$\leq\begin{cases}\dfrac{|b-a|^3}{160}\left\|f'''\right\|_{[a,b],\infty} & \text{if } f'''\in L_\infty[a,b];\\[2ex] \dfrac{\beta^2|b-a|^{2+\frac{1}{\beta}}}{4(3\beta+1)(4\beta+1)(\beta+1)^{\frac{1}{\beta}}}\left\|f'''\right\|_{[a,b],\alpha} & \text{if } f'''\in L_\alpha[a,b],\\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \dfrac{(b-a)^2}{48}\left\|f'''\right\|_{[a,b],1}, & \end{cases}$$

for either $a<b$ or $b<a$.

If we put in (5.5) $a=1$ and $b=\frac{q(x)}{p(x)}$, we get

$$\left|\frac{\int_1^{\frac{q(x)}{p(x)}}f(t)\,dt}{\frac{q(x)}{p(x)}-1}-\frac{1}{2}f\left(\frac{q(x)}{p(x)}\right)+\frac{q(x)-p(x)}{12p(x)}\left[f'\left(\frac{q(x)}{p(x)}\right)-f'(1)\right]\right| \tag{5.6}$$
$$\leq\begin{cases}\dfrac{|q(x)-p(x)|^3}{160p^3(x)}\left\|f'''\right\|_{\left[\frac{q(x)}{p(x)},1\right],\infty}\\[2ex] \dfrac{\beta^2|q(x)-p(x)|^{2+\frac{1}{\beta}}}{4(3\beta+1)(4\beta+1)(\beta+1)^{\frac{1}{\beta}}[p(x)]^{2+\frac{1}{\beta}}}\left\|f'''\right\|_{\left[\frac{q(x)}{p(x)},1\right],\alpha}\\[2ex] \dfrac{(q(x)-p(x))^2}{48p^2(x)}\left\|f'''\right\|_{\left[\frac{q(x)}{p(x)},1\right],1}\end{cases}$$
$$\leq\begin{cases}\dfrac{|q(x)-p(x)|^3}{160p^3(x)}\left\|f'''\right\|_{[r,R],\infty}\\[2ex] \dfrac{\beta^2|q(x)-p(x)|^{2+\frac{1}{\beta}}}{4(3\beta+1)(4\beta+1)(\beta+1)^{\frac{1}{\beta}}[p(x)]^{2+\frac{1}{\beta}}}\left\|f'''\right\|_{[r,R],\alpha}\\[2ex] \dfrac{(q(x)-p(x))^2}{48p^2(x)}\left\|f'''\right\|_{[r,R],1}.\end{cases}$$

If we multiply (5.6) by $p(x)\geq 0$, integrate on χ and take into consideration that

$$\int_\chi p(x)\,d\mu(x)=\int_\chi q(x)\,d\mu(x)=1,$$

then we get (5.4). ∎

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