ON PERTURBED TRAPEZOIDAL AND MIDPOINT RULES

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ABSTRACT. Explicit bounds are obtained for the perturbed, or corrected, trapezoidal and midpoint rules in terms of the Lebesque norms of the second derivative of the function. It is demonstrated that the bounds obtained are the same for both rules although the perturbation or the correction term is different.

1. INTRODUCTION

Let $f : [a, b] \to \mathbb{R}$ and define the functionals

$$I(f) := \int_a^b f(t)\, dt \tag{1.1}$$

$$I^{(T)}(f) := \frac{b-a}{2}[f(a) + f(b)] \tag{1.2}$$

and

$$I^{(M)}(f) := (b-a) f\left(\frac{a+b}{2}\right). \tag{1.3}$$

Here $I^{(T)}(f)$ and $I^{(M)}(f)$ are the well known trapezoidal and midpoint rules used to approximate the functional $I(f)$.

Atkinson [1] defined the corrected or perturbed trapezoidal and midpoint rules by

$$PI^{(T)}(f) := I^{(T)}(f) - \frac{c^2}{3}[f'(b) - f'(a)] \tag{1.4}$$

and

$$PI^{(M)}(f) := I^{(M)}(f) + \frac{c^2}{6}[f'(b) - f'(a)] \tag{1.5}$$

respectively, where $c = \frac{b-a}{2}$.

Atkinson [1] uses an asymptotic error estimate which does not readily produce estimates of the bounds in using (1.4) and (1.5) to approximate (1.1) by (1.4) or (1.5). In a recent article Barnett and Dragomir [2] obtained explicit bounds for $\left|I(f) - PI^{(T)}(f)\right|$ in terms of the Lebesque norms of $f''(t) - [f'; a, b]$ where $[f'; a, b] := \frac{f'(b)-f'(a)}{b-a}$ is the divided difference. If the Lebesque norms are defined in the usual way such that by $h \in L_p[a, b]$ we mean

$$\|h\|_p := \left(\int_a^b |h(t)|^p\, dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty \tag{1.6}$$

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and

$$\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|. \tag{1.7}$$

Barnett and Dragomir [2] obtained the following theorem.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable and f' absolutely continuous on $[a,b]$ then*

(1.8)
$$\left|I(f)-PI^{(T)}(f)\right| \le \begin{cases} \dfrac{(b-a)^3}{24}\left\|f''-[f';a,b]\right\|_\infty & \text{if } f''\in L_\infty[a,b]; \\[2ex] \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f''-[f';a,b]\right\|_p, & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{(b-a)^2}{8}\left\|f''-[f';a,b]\right\|_1, \end{cases}$$

where $[h;a,b]:=\frac{h(b)-h(a)}{b-a}$ is the divided difference.

Further, the Grüss integral inequality for a function $h:[a,b]\to\mathbb{R}$ with $-\infty<m\le h(x)\le m<\infty$ for almost every $x\in[a,b]$ then (see for example [10, p. 296])

$$0\le \frac{1}{b-a}\|h\|_2-\mathcal{M}^2(h)\le\left(\frac{M-m}{2}\right)^2, \tag{1.9}$$

where $\mathcal{M}(h)=\frac{1}{b-a}\int_a^b h(t)\,dt$, is the integral mean.

Barnett and Dragomir [2] also obtained the following results.

Let $f:[a,b]\to\mathbb{R}$, then

$$\left|I(f)-PI^{(T)}(f)\right|\le\frac{(b-a)^3}{8\sqrt{5}}\left[\frac{1}{b-a}\|f''\|_2^2-[f';a,b]^2\right]^{\frac{1}{2}},\quad f''\in L_2[a,b] \tag{1.10}$$

and

$$\left|I(f)-PI^{(T)}(f)\right|\le\frac{(b-a)^3}{16\sqrt{5}}(\Gamma-\gamma),\qquad \gamma\le f''(t)\le\Gamma \text{ a.e. } t\in[a,b]. \tag{1.11}$$

Result (1.10) is obtained from the second inequality in (1.8) and (1.11) from (1.9) and (1.10).

It is the intention of the current article to demonstrate that bounds for $\left|I(f)-PI^{(T)}(f)\right|$ may be obtained involving the traditional Lebesque norms of $\|f''\|_p$, $p\ge1$ where $\|\cdot\|_p$ are as defined by (1.6) and (1.7) rather than $\|f''-[f';a,b]\|_p$ as obtained in (1.8). Further, bounds will be obtained for $\left|I(f)-PI^{(M)}(f)\right|$. These will be shown to be the same as those obtained for the perturbed trapezoidal rule although the correction term is different.

2. Identities and Inequalities for Trapezoidal Like Rules

Let the trapezoidal functional $T(f;a,b)$ be defined by

$$T(f;a,b):=I(f)-I^{(T)}(f)=\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{b-a}{2}[f(a)+f(b)] \tag{2.1}$$

then it is well known that the identity

$$T(f;a,b) = -\frac{1}{2}\int_a^b (t-a)(b-t) f''(t)\,dt \tag{2.2}$$

holds. The following theorem was obtained in [9] using identity (2.2).

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b). Then we have the estimate*

$$|T(f;a,b)| \le \begin{cases} \frac{\|f''\|_\infty}{12}(b-a)^3 & \text{if } f''\in L_\infty[a,b]; \\ \frac{1}{2}\|f''\|_q [B(q+1,q+1)]^{\frac{1}{p}} (b-a)^{2+\frac{1}{p}}, & \text{if } f''\in L_p[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1 \\ \frac{\|f''\|_1}{8}(b-a)^2 & \text{if } f''\in L_1[a,b], \end{cases} \tag{2.3}$$

where B is the Beta function, that is,

$$B(r,s) := \int_0^1 t^{r-1}(1-t)^{s-1}\,dt,\ r,s>0.$$

Let

$$PT(f;a,b) := I(f) - PI^{(T)}(f) = T(f;a,b) + \frac{c^2}{3}[f'(b)-f'(a)], \tag{2.4}$$

where $c=\frac{b-a}{2}$, then the following lemma holds.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$ then*

$$PT(f;a,b) = \frac{1}{2}\int_a^b \kappa(t) f''(t)\,dt \tag{2.5}$$

is valid with

$$\kappa(t) = \left(t-\frac{a+b}{2}\right)^2 - \frac{c^2}{3},\quad c=\frac{b-a}{2} \tag{2.6}$$

Proof. From (2.4) and (2.6) we have

$$\begin{aligned} PT(f;a,b) &= -\frac{1}{2}\int_a^b (t-a)(b-t)f''(t)\,dt + \frac{c^2}{3}[f'(b)-f'(a)] \\ &= \int_a^b \left[\frac{c^2}{3} - \frac{1}{2}(t-a)(b-t)\right] f''(t)\,dt \\ &= \frac{1}{2}\int_a^b \left[t^2-(a+b)t+ab+2\frac{c^2}{3}\right] f''(t)\,dt \\ &= \frac{1}{2}\int_a^b \left[\left(t-\frac{a+b}{2}\right)^2 - \frac{c^2}{3}\right] f''(t)\,dt \end{aligned}$$

and so (2.5) holds with $\kappa(t)$ as given by (2.6). ∎

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$ then*

(2.7)
$$|PT(f;a,b)| \leq \begin{cases} \frac{4c^3}{9\sqrt{3}}\|f''\|_\infty, & \text{if } f''\in L_\infty[a,b]; \\ \frac{c^2}{6}\left\{\frac{c}{\sqrt{3}}B\left(\frac{1}{2},q+1\right)+2\int_1^{\sqrt{3}}\left(u^2-1\right)^q du\right\}^{\frac{1}{q}}\|f''\|_p, & \text{if } f''\in L_p[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1 \\ \frac{c^2}{6}\|f''\|_1 & \text{if } f''\in L_1[a,b], \end{cases}$$

where B is the beta function and $c=\frac{b-a}{2}$.

Proof. From identity (2.5) and using (2.6) we have

$$|PT(f;a,b)| \leq \frac{1}{2}\int_a^b |\kappa(t)||f''(t)|\,dt \leq \frac{\|f''\|_p}{2}\left(\int_a^b |\kappa(t)|^q\,dt^{\frac{1}{q}}\right),\ p>1. \tag{2.8}$$

Now we need to examine the behaviour of $\kappa(t)$ in order to proceed further. We notice from (2.6) that $\kappa(a)=\kappa(b)=\frac{2}{3}c^2$ and $\kappa(t)=0$ where $t=\frac{a+b}{2}\pm\frac{c}{\sqrt{3}}$.

Further,

$$\kappa'(t)=2\left(t-\frac{a+b}{2}\right)\begin{cases} <0, & t<\frac{a+b}{2}; \\ =0, & t=\frac{a+b}{2}; \\ >0, & t>\frac{a+b}{2}. \end{cases}$$

Also, $\kappa(t)$ is a symmetric function about $\frac{a+b}{2}$ since $\kappa\left(\frac{a+b}{2}+x\right)=\kappa\left(\frac{a+b}{2}-x\right)$ so that from (2.8)

$$\begin{aligned} \|\kappa\|_q^q &= \int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{c}{\sqrt{3}}}[-\kappa(t)]^q\,dt+\int_{\frac{a+b}{2}+\frac{c}{\sqrt{3}}}^b \kappa^q(t)\,dt \\ : &= 2\left[I_1(q)+I_2(q)\right]. \end{aligned} \tag{2.9}$$

Further, from (2.6)

$$I_1(q)=\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{c}{\sqrt{3}}}\left[\frac{c^2}{3}-\left(t-\frac{a+b}{2}\right)^2\right]^q dt$$

Let $\frac{c}{\sqrt{3}}u=t-\frac{a+b}{2}$, then

$$I_1(q)=\frac{c}{\sqrt{3}}\int_0^1\left(\frac{c^2}{3}\right)^q\left(1-u^2\right)^q du=\frac{c^{2q+1}}{3^{q+\frac{1}{2}}}\cdot\frac{1}{2}B\left(\frac{1}{2},q+1\right) \tag{2.10}$$

since $\int_0^1\left(1-u^2\right)^q du=\frac{1}{2}B\left(\frac{1}{2},q+1\right)$.

Also,

$$I_2(q)\int_{\frac{a+b}{2}+\frac{c}{\sqrt{3}}}^b\left[\left(t-\frac{a+b}{2}\right)^2-\frac{c^2}{3}\right]^q dt$$

and substituting $\frac{c}{\sqrt{3}}v = t - \frac{a+b}{2}$ gives

$$I_2(q) = \frac{c}{\sqrt{3}}\left(\frac{c^2}{3}\right)^q \int_0^1 \left[v^2 - 1\right]^q dv. \tag{2.11}$$

Combining (2.10) and (2.11) into (2.9) gives from (2.8) the second inequality in (2.7).

The first inequality is obtained by taking $q = 1$ in the second inequality of (2.7) as may be noticed from (2.8).

Thus

$$\begin{aligned} \frac{1}{2}\int_a^b |\kappa(t)|\, dt &= \frac{c^3}{6\sqrt{3}}\left[B\left(\frac{1}{2}, 2\right) + 2\int_1^{\sqrt{3}} \left(u^2 - 1\right) du\right] \\ &= \frac{c^3}{6\sqrt{3}}\left[\frac{4}{3} + \frac{4}{3}\right] = \frac{4c^3}{3^{\frac{5}{2}}}. \end{aligned}$$

Now, for the final inequality, from (2.8) we obtain

$$|PT(f; a, b)| \le \frac{1}{2}\sup_{t\in[a,b]} |\kappa(t)| \, \|f''\|_1$$

and so from the behaviour of $\kappa(t)$ discussed earlier

$$\sup_{t\in[a,b]} |\kappa(t)| = \max\left\{\frac{2}{3}c^2, \frac{1}{3}c^2\right\} = \frac{2}{3}c^2.$$

∎

The following corollary involving the Euclidean norm is of particular importance.

Corollary 1. *Let $f;[a,b] \to \mathbb{R}$ be such that $f'' \in L_2[a,b]$. Then we have the inequality*

$$|PT(f; a, b)| \le \frac{\sqrt{2}c^{\frac{5}{2}}}{3\sqrt{5}} \|f''\|_2 = \frac{(b-a)^{\frac{5}{2}}}{6\sqrt{5}} \|f''\|_2, \tag{2.12}$$

where $c = \frac{b-a}{2}$.

Proof. Taking $p = q = 2$ in (2.7) gives

$$|PT(f; a, b)| \le \frac{c^2}{6}\left\{\frac{c}{\sqrt{3}}\left[B\left(\frac{1}{2}, 3\right) + 2\int_1^{\sqrt{3}} \left(u^2 - 1\right)^2 du\right]\right\}^{\frac{1}{2}} \|f''\|_2$$

which, upon using the facts that

$$B\left(\frac{1}{2}, 3\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3)}{\Gamma\left(\frac{7}{2}\right)} = \frac{16}{15}$$

and

$$\int_1^{\sqrt{3}} \left(u^2 - 1\right)^2 du = 4\left(\frac{3\sqrt{3} - 2}{15}\right)$$

gives the stated result (2.12) after some simplification. ∎

3. Identities and Inequalities for Midpoint Like Rules

Let, from (1.1) and (1.3) the midpoint functional, $M(f;a,b)$ be defined by

$$M(f;a,b):=I(f)-I^{(M)}(f)=\frac{1}{b-a}\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right) \tag{3.1}$$

then the identity

$$M(f;a,b)=\int_a^b \phi(t)\,f''(t)\,dt \tag{3.2}$$

is well known, where

$$\phi(t)=\begin{cases}\dfrac{(t-a)^2}{2}, & t\in\left[a,\dfrac{a+b}{2}\right],\\[2ex] \dfrac{(b-t)^2}{2}, & t\in\left(\dfrac{a+b}{2},b\right].\end{cases} \tag{3.3}$$

The following theorem concerning the classical midpoint functional (3.1) with bounds involving the $L_p[a,b]$ norms of the second derivative is known (see [8] and [6]).

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$. Then*

$$|M(f;a,b)|\le\begin{cases}\frac{(b-a)^3}{24}\left\|f''\right\|_\infty & \text{if } f''\in L_\infty[a,b];\\[2ex] \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f''\right\|_p, & \text{if } f''\in L_p[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1\\ \frac{(b-a)^2}{8}\left\|f''\right\|_1 & \text{if } f''\in L_1[a,b],\end{cases} \tag{3.4}$$

The first inequality in (3.4) is the one that is traditionally most well known. Further, from (1.5) and (3.1) define the perturbed or corrected midpoint functional as

$$PM(f;a,b):=I(f)-PI^{(M)}(f)=M(f;a,b)-\frac{c^2}{6}\left[f'(b)-f'(a)\right], \tag{3.5}$$

where $c=\frac{b-a}{2}$.

The following lemma concerning $PM(f;a,b)$ holds.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$. Then*

$$PM(f;a,b)=\frac{1}{2}\int_a^b \chi(t)\,f''(t)\,dt, \tag{3.6}$$

where

$$\chi(t)=\begin{cases}(t-a)^2-\frac{1}{3}c^2, & t\in\left[a,\dfrac{a+b}{2}\right],\\[2ex] (b-t)^2-\frac{1}{3}c^2, & t\in\left(\dfrac{a+b}{2},b\right].\end{cases} \tag{3.7}$$

with $c=\frac{b-a}{2}$.

Proof. From (3.5) and (3.7) we have, on utilising (3.2),

$$\begin{aligned} PM(f;a,b) &= M(f;a,b) - \frac{c^2}{6}\left[f'(b) - f'(a)\right] \\ &= \int_a^b \left[\phi(t) - \frac{c^2}{6}\right] f''(t)\,dt \\ &= \frac{1}{2}\int_a^b \left[2\phi(t) - \frac{c^2}{3}\right] f''(t)\,dt. \end{aligned}$$

Now, from the definition of $\phi(t)$ from (3.3) gives

$$2\phi(t) - \frac{c^2}{3} = \begin{cases} (t-a)^2 - \dfrac{c^2}{3}, & t \in \left[a, \dfrac{a+b}{2}\right], \\ (b-t)^2 - \dfrac{c^2}{3}, & t \in \left(\dfrac{a+b}{2}, b\right]. \end{cases}$$

and the result as stated in (3.6) readily follows and the lemma is thus proved. ∎

Theorem 5. *The Lebesgue norms for the perturbed midpoint functional $PM(f;a,b)$ as given by (3.6) are the same as those for the perturbed trapezoid function $PT(f;a,b)$ given by (2.7).*

Proof. To prove the theorem it suffices to demonstrate that

$$\|\chi\|_p = \|\kappa\|_p, \quad p \geq 1. \tag{3.8}$$

The properties of κ were investigated in the proof of Theorem 3. Now for $\chi(t)$.

$\chi(a) = \chi(b) = -\frac{c^2}{3}$ and $\chi(t) = 0$ when $t = a + \frac{c}{\sqrt{3}}, b - \frac{c}{\sqrt{3}}$ for $t \in [a,b]$. Further, $\chi(t)$ is continuous at $t = \frac{a+b}{2}$ and $\chi\left(\frac{a+b}{2}\right) = \frac{2}{3}c^2$. Also

$$\chi'(t) = \begin{cases} 2(t-a) > 0, & t \in \left[a, \dfrac{a+b}{2}\right], \\ -2(b-t) < 0, & t \in \left(\dfrac{a+b}{2}, b\right]. \end{cases}$$

As a matter of fact, for $t \in \left[a, \frac{a+b}{2}\right]$, $\chi(t) = \kappa\left(a + \frac{a+b}{2} - t\right)$ and for $t \in \left(\frac{a+b}{2}, b\right]$, $\chi(t) = \kappa\left(b + \frac{a+b}{2} - t\right)$. That is, $\chi(t)$ and $\kappa(t)$ are symmetric about $\frac{3a+b}{4}$, the midpoint of $\left[a, \frac{a+b}{2}\right]$ and $\frac{a+3b}{4}$ the midpoint of $\left(\frac{a+b}{2}, b\right]$. Thus, (3.8) holds and the theorem is valid as stated. ∎

Remark 1. *The bound given in (2.12) also holds for $PM(f;a,b)$ given the results of Theorem 4.*

4. Perturbed Rules from the Chebychev Functional

For $g, h : [a,b] \to \mathbb{R}$ the following $\mathfrak{T}(g,h)$ is well known as the Chebychev functional. Namely,

$$\mathfrak{T}(g,h) = \mathcal{M}(gh) - \mathcal{M}(g)\,\mathcal{M}(h), \tag{4.1}$$

where $\mathcal{M}(g) = \frac{1}{b-a}\int_a^b g(t)\,dt$ is the integral mean.

The Chebychev functional (4.1) is known to satisfy a number of identities including

$$\mathfrak{T}(g,h)=\frac{1}{b-a}\int_a^b h(t)\left[g(t)-\mathcal{M}(g)\right]dt. \tag{4.2}$$

Further, a number of sharp bounds for $|\mathfrak{T}(g,h)|$ exist, under various assumptions about g and h, including (see [5] for example)

$$|\mathfrak{T}(g,h)|\le\begin{cases}\left[\mathfrak{T}(g,g)\right]^{\frac{1}{2}}\left[\mathfrak{T}(h,h)\right]^{\frac{1}{2}}, & g,h\in L_2[a,b]\\[2mm] \dfrac{A_u-A_l}{2}\left[\mathfrak{T}(h,h)\right]^{\frac{1}{2}}, & A_l\le g(t)\le A_u,\ t\in[a,b]\\[2mm] \left(\dfrac{A_u-A_l}{2}\right)\left(\dfrac{B_u-B_l}{2}\right), & B_l\le h(t)\le B_u,\ t\in[a,b]\ \text{(Grüss)}.\end{cases} \tag{4.3}$$

It will be demonstrated how (4.1) may be used to obtain perturbed results which (4.2) will provide an identity with which to obtain bounds. The following theorem holds.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous, then*

$$\begin{aligned}|PT(f;a,b)| &\le \frac{(b-a)^3}{12\sqrt{5}}\left[\frac{1}{b-a}\|f''\|_2^2-[f';a,b]^2\right]^{\frac{1}{2}}, \qquad f''\in L_2[a,b]\\ &\le \frac{(b-a)^3}{24\sqrt{5}}(B_u-B_l),\ \ B_l\le f''(t)\le B_u,\ t\in[a,b],\end{aligned} \tag{4.4}$$

where $PT(f;a,b)$ is the perturbed trapezoidal rule defined by (2.4).

Proof. Let $g(t)=-\frac{1}{2}(t-a)(b-t)$, the trapezoidal kernel and $h(t)=f''(t)$ then from (4.1)

$$\begin{aligned}(b-a)\,\mathfrak{T}(g(t),f''(t)) &= \int_a^b g(t)f''(t)\,dt-\mathcal{M}(g)\int_a^b f''(t)\,dt\\ &= T(f;a,b)+\frac{c^2}{3}\left[f'(b)-f'(a)\right],\end{aligned} \tag{4.5}$$

where $\mathcal{M}(g)=-\frac{c^2}{3}$.

Now, from (4.2)

$$\begin{aligned}(b-a)\,\mathfrak{T}(g(t),f''(t)) &= \int_a^b f''(t)\left[g(t)+\frac{c^2}{3}\right]dt\\ &= \frac{1}{2}\int_a^b \kappa(t)f''(t)\,dt\end{aligned} \tag{4.6}$$

and so (4.4) and (4.5) produce identities (2.5) – (2.6).

Thus, from (4.3) and (4.4) we get

(4.7)

$$|PT(f;a,b)| \le \begin{cases} \left[(b-a)\mathfrak{T}(g,g)\right]^{\frac{1}{2}}\left[(b-a)\mathfrak{T}(f'',f'')\right]^{\frac{1}{2}}, & g,h\in L_2[a,b] \\ \left(\dfrac{A_u-A_l}{2}\right)\left[(b-a)\mathfrak{T}(f'',f'')\right]^{\frac{1}{2}}, & A_l\le g(t)\le A_u, \\ \left(\dfrac{A_u-A_l}{2}\right)\left(\dfrac{B_u-B_l}{2}\right)(b-a), & B_l\le f''(t)\le B_u. \end{cases}$$

Here, from (4.1)

$$\begin{aligned} \left[(b-a)\mathfrak{T}(h,h)\right]^{\frac{1}{2}} &= \left[\int_a^b h^2(t)\,dt-(b-a)\mathcal{M}^2(h)\right]^{\frac{1}{2}} \\ &= (b-a)^{\frac{1}{2}}\left[\frac{1}{b-a}\|h\|_2^2-\mathcal{M}^2(h)\right]^{\frac{1}{2}}. \end{aligned}$$

Specifically,

$$\begin{aligned} \text{(4.8)}\quad \left[(b-a)\mathfrak{T}(f'',f'')\right]^{\frac{1}{2}} &= (b-a)^{\frac{1}{2}}\left[\frac{1}{b-a}\|f''\|_2^2-\left[\frac{f'(b)-f'(a)}{b-a}\right]^2\right]^{\frac{1}{2}} \\ &= (b-a)^{\frac{1}{2}}\left[\frac{1}{b-a}\|f''\|_2^2-[f;a,b]^2\right]^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \text{(4.9)}\quad & \left[(b-a)\mathfrak{T}(g(t),g(t))\right]^{\frac{1}{2}} \\ &= \left\{\int_a^b g^2(t)\,dt-\left(\frac{\int_a^b g(t)\,dt}{b-a}\right)^2\right\}^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{1}{2}}}{2}\left[\frac{1}{b-a}\int_a^b (t-a)^2(b-t)^2\,dt-\left(\frac{\int_a^b (t-a)(b-t)\,dt}{b-a}\right)^2\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{5}{2}}}{2}\left[B(3,3)-B^2(2,2)\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{5}{2}}}{2}\left[\frac{(2!)^2}{5!}-\left(\frac{1}{3!}\right)^2\right]^{\frac{1}{2}} = \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}. \end{aligned}$$

Utilising (4.8) and (4.9) into the first result in (4.7) gives the first result (4.4). For the second result in (4.4) we utilise (1.9) giving the stated coarser bound. ∎

Remark 2. *Even though* $A_l = -\frac{c^2}{2} \le g(t) \le 0 = A_u$, *it is not worthwhile using this in the second and third inequality of (4.5) as this would produce a coarser bound than those stated in Theorem 6.*

Remark 3. *The results of Theorem 6 as represented by (4.4) are tighter than (1.10) and (1.11). For a different proof of the sharpness of (4.4) see* [3].

The following bounds for the perturbed midpoint rule holds.

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous, then*

$$\begin{aligned}\left|PM\left(f;a,b\right)\right| &\leq \frac{\left(b-a\right)^{3}}{12\sqrt{5}}\left[\frac{1}{b-a}\left\|f''\right\|_{2}^{2}-\left[f';a,b\right]^{2}\right]^{\frac{1}{2}}, \quad f''\in L_{2}\left[a,b\right] \\ &\leq \frac{\left(b-a\right)^{3}}{24\sqrt{5}}\left(B_{u}-B_{l}\right), \;\; B_{l}\leq f''\left(t\right)\leq B_{u},\; t\in\left[a,b\right].\end{aligned}$$

Proof. The proof follows readily from Theorem 5 since

$$\mathfrak{T}\left(\phi,\phi\right)=\frac{1}{2}\left\|\chi\right\|_{2},$$

where $\phi\left(t\right)$ and $\chi\left(t\right)$ are as given by (3.3) and (3.7) respectively. ∎

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