ON RELATIONSHIPS BETWEEN OSTROWSKI, TRAPEZOIDAL AND CHEBYCHEV IDENTITIES AND INEQUALITIES

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Abstract. This article investigates the relationship between the Ostrowski, Trapezoidal and Chebychev functional identities. It demonstrates how one may be obtained from the other through transformations. The relationships between bounds on the functionals is also examined. The three point functional is analysed as is its relationship to a generalised Chebyshev functional.

1. Introduction

In the paper we investigate the relationships between the three functionals, for $f : [a,b] \to \mathbb{R}$,

$$S(f;a,x,b) := f(x) - \mathcal{M}(f;a,b), \tag{1.1}$$

$$T(f;a,x,b) := \left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) - \mathcal{M}(f;a,b), \tag{1.2}$$

and

$$\mathfrak{T}(f,g;a,b) := \mathcal{M}(fg;a,b) - \mathcal{M}(f;a,b)\,\mathcal{M}(g;a,b), \tag{1.3}$$

where

$$\mathcal{M}(f;a,b) := \frac{1}{b-a}\int_a^b f(u)\,du, \qquad \text{the integral mean.} \tag{1.4}$$

The above functionals will be termed as the Ostrowski, Trapezoidal and Chebychev functionals respectively for reasons that will be fairly evident subsequently.

We note firstly that

$$(b-a)\,S\left(f;a,\frac{a+b}{2},b\right) = (b-a)\,f\left(\frac{a+b}{2}\right) - \int_a^b f(u)\,du \tag{1.5}$$

and

$$(b-a)\,T\left(f;a,\frac{a+b}{2},b\right) = \frac{b-a}{2}\left[f(a)+f(b)\right] - \int_a^b f(u)\,du, \tag{1.6}$$

recapturing the midpoint and trapezoidal rules for the evaluation of the integrals. With this in mind, the most common task is to obtain bounds on the functionals (1.1) – (1.3). This task is perhaps best accomplished from identities involving the functionals.

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The approach will be demonstrated on the functional $S(f;a,x,b)$. It may easily be demonstrated through an integration by parts argument of the Riemann-Stieltjes integral on the intervals $[a,x]$ and $(x,b]$ that the identity [10]

$$S(f;a,x,b)=\int_a^b p(x,t)\,df(t),\quad p(x,t)=\begin{cases}\dfrac{t-a}{b-a}, & t\in[a,x],\\[2ex] \dfrac{t-b}{b-a}, & t\in(x,b]\end{cases} \tag{1.7}$$

holds. For f absolutely continuous then $df(t)\equiv f'(t)\,dt$ and (1.7) becomes the Montgomery identity. The following theorem may be proved using the Montgomery identity (see Dragomir and Wang [14] – [16]).

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$, we have:*

$$|S(f;a,x,b)| \tag{1.8}$$

$$\leq\begin{cases}\left[\dfrac{1}{4}+\left(\dfrac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if}\quad f'\in L_\infty[a,b];\\[3ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\dfrac{x-a}{b-a}\right)^{p+1}+\left(\dfrac{b-x}{b-a}\right)^{p+1}\right](b-a)^{\frac{1}{p}}\|f'\|_q & \text{if}\quad f'\in L_q[a,b],\\ & \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\[2ex] \left[\dfrac{1}{2}+\left|\dfrac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1;\end{cases} \tag{1.9}$$

where $S(f;a,x,b)$ is as given by (1.1), $\|\cdot\|_r$ $(r\in[1,\infty])$ are the usual Lebesque norms on $L_r[a,b]$, namely,

$$\|g\|_\infty:=ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_r:=\left(\int_a^b|g(t)|^r\,dt\right)^{\frac{1}{r}},\qquad r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense that they cannot be replaced by a smaller constant.

Ostrowski [19] proved the first inequality in (1.8) in 1938, using a different argument and hence the justification for the naming of $S(f;a,x,b)$. Fink [17] also obtained generalisations of the above results.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [13]):

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, that is,*

$$|f(x)-f(y)|\leq H|x-y|^r,\qquad \textit{for all } x,y\in[a,b],$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then, for all $x \in [a,b]$, we have the inequality:

$$|S(f;a,x,b)| \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r, \tag{1.10}$$

where $S(f;a,x,b)$ is as given by (1.1).
The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, that is f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [12])

$$|S(f;a,x,b)| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.11}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [10]).

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then*

$$|S(f;a,x,b)| \leq \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.12}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, that is, f is monotonically increasing, then the inequality (1.12) may be improved in the following manner [11].

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a,b]$, we have the inequalities:*

$$\begin{aligned} &|S(f;a,x,b)| \\ &\leq \frac{1}{b-a}\left\{[2x-(a+b)]f(x) + \int_a^b sgn(t-x)f(t)\,dt\right\} \\ &\leq \frac{1}{b-a}\{(x-a)[f(x)-f(a)] + (b-x)[f(b)-f(x)]\} \\ &\leq \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)], \end{aligned} \tag{1.13}$$

where $S(f;a,x,b)$ is as given by (1.1).
All the inequalities in (1.13) are sharp and the constant $\frac{1}{2}$ is the best possible.

The interested reader is encouraged to see [6] for an extensive treatment of the Ostrowski (interior point) rule and its applications to numerical quadrature. Also, for other recent results including Ostrowski type inequalities for $n-$time differentiable functions, visit the RGMIA website at `http://rgmia.vu.edu.au/database.html`.

It may further be demonstrated by integration by parts of the Riemann-Stieltjes integral that the identity [7]

$$T(f;a,x,b) = \int_a^b q(x,t)\,df(t), \quad q(x,t) = \frac{t-x}{b-a}, \ \ x,t \in [a,b], \tag{1.14}$$

holds. Cerone, Dragomir and Pearce [7] used the identity (1.14) and showed its applicability to numerical integration, special means and probability. They showed that for $f:[a,b]\to\mathbb{R}$ of bounded variation

$$|T(f;a,x,b)|\leq\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.15}$$

for all $x\in[a,b]$ and $\frac{1}{2}$ is best possible where $\bigvee_a^b(f)$ is the total variation of f.

More recently the author [2] showed that

$$\mathfrak{T}(f,g;a,b)=\frac{1}{(b-a)^2}\int_a^b\psi(t)\,df(t), \tag{1.16}$$

where

$$\psi(t)=(t-a)\,G(t,b)-(b-t)\,G(a,t),\quad G(a,b)=\int_a^b g(u)\,du. \tag{1.17}$$

For many other identities for the Chebychev functional $\mathfrak{T}(f,g;a,b)$ the interested reader is referred to [18] Chapters IX and I, 239 – 310.

In the paper [2] bounds were obtained on $\mathfrak{T}(f,g;a,b)$. Specifically, bounds for the moments and moment generating function were explicitly determined.

To elaborate, in the current paper, relationships between the functionals (1.1), (1.2) and (1.3) will be investigated through their respective identities (1.7), (1.14) and (1.16). It will be shown that one identity may be obtained from the other through some transformation. An investigation of the effect on their respective bounds is also undertaken.

In Section 3 a three point rule which was introduced in Cerone and Dragomir [5] will be examined and a generalised Chebychev functional is obtained following the techniques developed in Section 2. The bounds for a generalised Ostrowski type identity for $n-$time differentiable functions is shown to be the same as bounds for a generalised trapezoidal rule in Section 4. Some sufficient characteristics for kernels to have the same Lebesgue norms is also investigated in Section 4.

2. Relationships Between the Functionals

We start with the identity (1.7). Allowing $x\to a$ and $x\to b$ gives

$$S(f;a,a,b)=\int_a^b p(a,t)\,df(t)$$

and

$$S(f;a,b,b)=\int_a^b p(b,t)\,df(t)$$

respectively. Thus,

$$\begin{aligned}&(x-a)\,S(f;a,a,b)+(b-x)\,S(f;a,b,b)\\=&\int_a^b[(x-a)\,p(a,t)+(b-x)\,p(b,t)]\,df(t)\end{aligned} \tag{2.1}$$

but

$$\begin{aligned} & (x-a)\,p(a,t)+(b-x)\,p(b,t) \\ = & (x-a)\left(\frac{t-b}{b-a}\right)+(b-x)\left(\frac{t-a}{b-a}\right) \\ = & \frac{b(t-a)-a(t-b)}{b-a}-x\left(\frac{t-a+b-t}{b-a}\right) \\ = & t-x=(b-a)\,q(x,t) \end{aligned}$$

and dividing (2.1) by $b-a$ gives the Trapezoidal identity (1.14).

In a similar way, if we commence with the Trapezoidal identity (1.14) and allow $x \to a$ and $x \to b$ gives

$$T(f;a,a,b)=\int_a^b q(a,t)\,df(t)=\int_a^b\left(\frac{t-a}{b-a}\right)df(t)$$

and

$$T(f;a,b,b)=\int_a^b q(b,t)\,df(t)=\int_a^b\left(\frac{t-b}{b-a}\right)df(t)$$

respectively, so that

$$(x-a)\,T(f;a,a,x)+(b-x)\,T(f;x,b,b)=\int_a^x (t-a)\,df(t)+\int_x^b (t-b)\,df(t)$$

and upon division by $(b-a)$ gives the identity (1.7).

It may be noticed that if $H(u)$ is the Heaviside unit function defined by

$$H(u):=\begin{cases} 1, & u>0 \\ 0, & u<0 \end{cases} \tag{2.2}$$

then $p(x,t)$ from (1.7) may be written in the following form

$$p(x,t)=\left(\frac{t-a}{b-a}\right)H(x-t)+\left(\frac{t-b}{b-a}\right)H(t-x), \quad x,t\in[a,b] \tag{2.3}$$

and let from (1.15)

$$q(x,t)=\hat{q}(x,t;a,b)=\left(\frac{t-x}{b-x}\right)H(t-a)\,H(b-t), \quad x\in[a,b]. \tag{2.4}$$

The following lemma holds.

Lemma 1. *For $p(x,t)$ defined by (2.3) and $q(x,t)$ by (1.15), then the following relationships are valid for $x,t\in[a,b]$*

$$q(x,t)=\frac{(x-a)\,p(a,t)+(b-x)\,p(b,t)}{b-a} \tag{2.5}$$

and

$$p(x,t)=\frac{(x-a)\,\hat{q}(a,t;a,x)+(b-x)\,\hat{q}(b,t;x,b)}{b-a}. \tag{2.6}$$

Proof. From (2.3), using (2.2), gives

$$p(a,t)=\frac{t-b}{b-a} \quad \text{and} \quad p(b,t)=\frac{t-a}{b-a}$$

and so

$$\frac{(x-a)\,p\,(a,t)+(b-x)\,p\,(b,t)}{b-a} = \frac{(x-a)\,(t-b)+(b-x)\,(t-a)}{(b-a)^2} = \frac{t-x}{b-a} = q\,(x,t)$$

producing identity (2.5).

Now to prove (2.6) use the more explicit representation of $q\,(x,t)$ as given by (2.4), then

$$\hat{q}\,(a,t;a,x) = \left(\frac{t-a}{x-a}\right) H\,(x-t) \quad \text{and} \quad \hat{q}\,(b,t;x,b) = \left(\frac{t-b}{b-x}\right) H\,(t-x)$$

and so

$$\begin{aligned} & \frac{(x-a)\,\hat{q}\,(a,t;a,x)+(b-x)\,\hat{q}\,(b,t;x,b)}{b-a} \\ = & \left(\frac{t-a}{b-a}\right) H\,(x-t) + \left(\frac{t-b}{b-a}\right) H\,(t-x) \\ = & \; p\,(x,t)\,. \end{aligned}$$

Hence the lemma is completely proved. ∎

Remark 1. *It should be emphasized that there is a strong relationship between the Ostrowski and trapezoidal functionals. This is highlighted by the symmetric transformations amongst their kernels that provide the identities (1.7) and (1.15). In particular, we note that the bound in (1.15) in the same as the bound in (1.12).*

Theorem 5. *The Lebesgue norms for the generalised trapezoidal rule and those for the Ostrowski functional are equal. That is,*

$$\|p\,(x,\cdot)\|_{\gamma} = \|q\,(x,\cdot)\|_{\gamma}\,, \quad \gamma \in [1,\infty]\,, \tag{2.7}$$

where $p\,(x,t)$ and $q\,(x,t)$ are as given in (1.7) and (1.15) or (2.3) and (2.4) respectively.

Proof. Let $\lambda = \frac{x-a}{b-a}$ and $1-\lambda = \frac{b-x}{b-a}$, then $|p\,(x,t)| = \frac{t-a}{b-a}$ varies linearly from 0 to λ, for $a \le t \le x$ and $|q\,(x,t)| = \frac{x-t}{b-a}$, varies linearly from λ to 0 for $a \le t \le x$. Further, $|p\,(x,t)|$ and $|q\,(x,t)|$ are symmetric about $t = \frac{a+x}{2}$ within the interval $t \in [a,x]$. Similarly, over the interval $t \in (x,b]$, $|p\,(x,t)| = \frac{b-t}{b-a}$ which varies linearly from 0 to $1-\lambda$ for $x < t \le b$ and $|q\,(x,t)| = \frac{t-x}{b-a}$ varies linearly from 0 to $1-\lambda$ for $t \in (x,b]$. Again $|p\,(x,t)|$ and $|q\,(x,t)|$ is symmetric about $t = \frac{x+b}{2}$, the midpoint of the interval $t \in (x,b]$. We thus conclude that (2.7) is true. ∎

Remark 2. *All the bounds obtained for the Ostrowski functional as given by (1.8) – (1.13) are valid for the Trapezoidal functional.*

We now illustrate the process for acquiring the identity (1.16) for the Chebychev functional from the identity (1.7) for the Ostrowski functional.

We start with (1.7) and multiplying by $\frac{g(x)}{b-a}$ and then integrating over $[a,b]$ to give, on assuming that the interchange of order is permissible,

$$\begin{aligned}
\mathfrak{T}(f,g;a,b) &= \frac{1}{b-a}\int_a^b\int_a^b g(x)\,p(x,t)\,df(t)\,dx \\
&= \frac{1}{b-a}\int_a^b\left[\int_a^x g(x)\left(\frac{t-a}{b-a}\right)+\int_x^b g(x)\left(\frac{t-b}{b-a}\right)\right]df(t)\,dx \\
&= \frac{1}{b-a}\int_a^b\left[\left(\frac{t-a}{b-a}\right)\int_t^b g(x)\,dx+\left(\frac{t-b}{b-a}\right)\int_a^t g(x)\,dx\right]df(t) \\
&= \frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t),
\end{aligned}$$

which is identity (1.16) where $\psi(t)$ is as given by (1.17).

Thus, we have noticed how multiplication and integration transforms the Ostrowski functional identity to the Chebychev functional identity.

Can we go the other way? The answer is in the affirmative.

From (1.16) and (1.17) let

$$g(t)=H(t-\alpha)\,H(\beta-t),\quad a\le\alpha<\beta\le b, \tag{2.8}$$

where $H(\cdot)$ is the Heaviside unit function defined by (2.2), to give

$$\begin{aligned}
\left(\frac{b-a}{\beta-\alpha}\right)\mathfrak{T}(f,g;a,b) &= D(f;a,\alpha,\beta,b):=\mathcal{M}(f;\alpha,\beta)-\mathcal{M}(f;a,b) \\
&= \frac{1}{(b-a)(\beta-\alpha)}\int_a^b\tilde{\psi}(t)\,df(t)
\end{aligned} \tag{2.9}$$

where $\tilde{\psi}(t)=\psi(t)$ with $g(t)$ as given by (2.8) and so from (1.17)

$$\tilde{\psi}(t)=\begin{cases}
(\beta-\alpha)(t-a), & a\le t\le\alpha; \\[2ex]
\alpha(b-a)-a(\beta-\alpha)-[(b-a)-(\beta-\alpha)]\,t, & \alpha<t<\beta; \\[2ex]
(\beta-\alpha)(t-b), & \beta\le t\le b.
\end{cases} \tag{2.10}$$

Now, if we take $\alpha=x$ and $\beta=x+h$ then from the left hand side of (2.9)

$$D(f;a,x,x+h,b)=\mathcal{M}(f;x,x+h)-\mathcal{M}(f;a,b)$$

giving

$$S(f;a,x,b)=\lim_{h\to0}D(f;a,x,x+h,b).$$

The right hand side of (2.9) and using (2.10) produces, for f continuous,

$$\begin{aligned}
&\lim_{h\to0}\left\{\frac{1}{b-a}\left[\int_a^x(t-a)\,df(t)+\frac{1}{h}\int_x^{x+h}[x(b-a)-ah-(b-a-h)\,t]\,df(t)\right.\right. \\
&\left.\left.+\int_{x+h}^b(t-b)\,df(t)\right]\right\} \\
&=\int_a^b p(x,t)\,df(t)
\end{aligned}$$

and so (1.7) is obtained.

For further work on various bounds for $D(f;a,\alpha,\beta,b)$ the difference between two means the reader is referred to Barnett et al. [1] and Cerone and Dragomir [3].

If in (2.9) and (2.10) we take $\alpha = a$ and $\beta = a+h$ and allow $h \to 0$, then we get after some simplification

$$f(a) - \mathcal{M}(f;a,b) = \int_a^b \left(\frac{t-b}{b-a}\right) df(t).$$

Further, taking $\alpha = b-h$ and $\beta = b$ gives

$$f(b) - \mathcal{M}(f;a,b) = \int_a^b \left(\frac{t-a}{b-a}\right) df(t)$$

from (2.9) and (2.10) after allowing $h \to 0$. Combining the above results gives

$$\begin{aligned} & \left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) \\ = & \frac{1}{(b-a)^2} \int_a^b [(x-a)(t-b) + (b-x)(t-a)] \, df(t) \\ = & \frac{1}{b-a} \int_a^b (t-x) \, df(t) \end{aligned}$$

and the identity for the Trapezoidal functional (1.14) is recaptured.

3. A Three Point Identity

Define the functional $\mathfrak{T}(f;a,\alpha,x,\beta,b)$ which involves the difference between the integral mean, a weighted combination of a function evaluated at the end points and an interior point. Namely, for $a \le \alpha < x < \beta \le b$,

$$\text{(3.1)} \qquad \begin{aligned} \mathfrak{T}(f;a,\alpha,x,\beta,b) \; : \; = & \left(\frac{\alpha-a}{b-a}\right) f(a) + \left(\frac{\beta-\alpha}{b-a}\right) f(x) \\ & + \left(\frac{b-\beta}{b-a}\right) f(b) - \mathcal{M}(f;a,b). \end{aligned}$$

Cerone and Dragomir [5] showed that the identity

$$\text{(3.2)} \qquad \mathfrak{T}(f;a,\alpha,x,\beta,b) = \int_a^b r(x,t) \, df(t), \quad r(x,t) = \begin{cases} \dfrac{t-\alpha}{b-a}, & t \in [a,x] \\ \\ \dfrac{t-\beta}{b-a}, & t \in (x,b] \end{cases}$$

is valid. They effectively demonstrated that the Ostrowski functional and the trapezoid functional could be recaptured as particular instances. Specifically, from (3.1) and (3.2),

$$S(f;a,x,b) = \mathfrak{T}(f;a,a,x,b,b)$$

and

$$T(f;a,x,b) = \mathfrak{T}(f;a,x,x,x,b),$$

where $S(f;a,x,b)$ and $T(f;a,x,b)$ are defined by (1.1) and (1.2) and satisfy identities (1.7) and (1.14) respectively.

To maintain the spirit of the current article, demonstrating that correspondences or transformations are bidirectional, it will now be shown that (3.2) may be obtained from (1.7) and (1.14). To accomplish this we extend the notation of the previous section and write (1.7) and (1.14) as

$$S(f;c,x,d)=\int_c^d p(x,t,c,d)\,df(t)\,,\quad p(x,t,c,d)=\begin{cases}\dfrac{t-c}{d-c}, & t\in[c,x]\\[2ex] \dfrac{t-d}{d-c}, & t\in(x,d]\end{cases} \tag{3.3}$$

and

$$T(f;c,x,d)=\int_c^d q(x,t,c,d)\,df(t)\,,\quad q(x,t,c,d)=\frac{t-x}{d-c},\ x,t\in[c,d]\,. \tag{3.4}$$

Now, from (3.3)

$$\begin{aligned}&\left(\frac{\alpha-a}{b-a}\right)S(f;a,a,\alpha)+\left(\frac{\beta-\alpha}{b-a}\right)S(f;\alpha,x,\beta)+\left(\frac{b-\beta}{b-a}\right)S(f;\beta,b,b) \\ =\ &\left(\frac{\alpha-a}{b-a}\right)\int_a^\alpha p(a,t,a,\alpha)\,df(t)+\left(\frac{\beta-\alpha}{b-a}\right)\int_\alpha^\beta p(x,t,\alpha,\beta)\,df(t) \\ &+\left(\frac{b-\beta}{b-a}\right)\int_\beta^b p(b,t,\beta,b)\,df(t)\end{aligned} \tag{3.5}$$

but

$$\begin{aligned}\left(\frac{\alpha-a}{b-a}\right)p(a,t,a,\alpha)&=\frac{t-\alpha}{b-a},\ \ t\in[a,\alpha]\,,\\ \left(\frac{\beta-\alpha}{b-a}\right)p(x,t,\alpha,\beta)&=\begin{cases}\dfrac{t-\alpha}{b-a}, & t\in(\alpha,x]\\[2ex] \dfrac{t-\beta}{b-a}, & t\in(x,\beta],\end{cases}\end{aligned}$$

and

$$\left(\frac{b-\beta}{b-a}\right)p(b,t,\beta,b)=\frac{t-\beta}{b-a},\ \ t\in(\beta,b]$$

which combined give $r(x,t)$. Hence the identity (3.2) is recovered since the left hand side is $\mathfrak{T}(f;a,\alpha,x,\beta,b)$.

Further, from (3.4)

$$\begin{aligned}&\left(\frac{x-a}{b-a}\right)T(f;a,\alpha,x)+\left(\frac{b-x}{b-a}\right)T(f;x,\beta,b)\\ =\ &\left(\frac{x-a}{b-a}\right)\int_a^x q(\alpha,t,a,x)\,df(t)+\left(\frac{b-x}{b-a}\right)\int_x^b q(\beta,t,x,b)\,df(t)\end{aligned}$$

but

$$\left(\frac{x-a}{b-a}\right)q(\alpha,t,a,x)=\frac{t-\alpha}{b-a},\ \ t\in[a,x]$$

and

$$\left(\frac{b-x}{b-a}\right)q(\beta,t,x,b)=\frac{t-\beta}{b-a},\ \ t\in(x,b]$$

giving the identity (3.2).

It has thus been demonstrated above that a combination of scaled Ostrowski and Trapezoidal identities produce the three point identity (3.4).

Remark 3. *We shall investigate Ostrowski's inequality for absolutely continuous functions as given by the first inequality in (1.8). This may be derived from the Montgomery identity*

$$S(f;a,x,b)=\int_a^b p(x,t,a,b)\,f'(t)\,dt, \tag{3.6}$$

where

$$p(x,t,a,b)=\begin{cases}\dfrac{t-a}{b-a}, & t\in[a,x],\\[2ex] \dfrac{t-b}{b-a}, & t\in(x,b].\end{cases} \tag{3.7}$$

Now,

$$\begin{aligned} S(f;a,x,b) &= S(f;a,x,x)+S(f;x,x,b) \\ &= \int_a^x p(x,t,a,x)\,f'(t)\,dt+\int_x^b p(x,t,x,b)\,f'(t)\,dt \\ &= \int_a^x \left(\frac{t-a}{b-a}\right) f'(t)\,dt+\int_x^b \left(\frac{t-b}{b-a}\right) f'(t)\,dt. \end{aligned} \tag{3.8}$$

Thus, by properties of the modulus and integral and the triangle inequality, we have

$$\begin{aligned} |S(f;a,x,b)| &\le \int_a^x \left(\frac{t-a}{b-a}\right)|f'(t)|\,dt+\int_x^b \left(\frac{b-t}{b-a}\right)|f'(t)|\,dt \\ &\le \left(\frac{1}{b-a}\int_0^{x-a} u\,du\right)\|f'\|_{\infty,[a,x]} \\ &\quad +\left(\frac{1}{b-a}\int_0^{b-x} u\,du\right)\|f'\|_{\infty,[x,b]} \\ &= \frac{(x-a)^2}{2(b-a)}\|f'\|_{\infty,[a,x]}+\frac{(b-x)^2}{2(b-a)}\|f'\|_{\infty,[x,b]} \\ &\le \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_{\infty,[a,b]} \\ &= \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,\|f'\|_{\infty,[a,b]}\,. \end{aligned} \tag{3.9}$$

We now investigate how the Ostrowski bounds may be used to obtain bounds for the three point rule.

From (3.4) the three point rule is given in terms of the Ostrowski rule and so

$$
\begin{aligned}
& \left|\mathfrak{T}(f;a,\alpha,x,\beta,b)\right| \\
= & \left|\left(\frac{\alpha-a}{b-a}\right)S(f;a,a,\alpha)+\left(\frac{\beta-\alpha}{b-a}\right)S(f;\alpha,x,\beta)+\left(\frac{b-\beta}{b-a}\right)S(f;\beta,b,b)\right| \\
\leq & \left(\frac{\alpha-a}{b-a}\right)\left|S(f;a,a,\alpha)\right|+\left(\frac{\beta-\alpha}{b-a}\right)\left|S(f;\alpha,x,\beta)\right|+\left(\frac{b-\beta}{b-a}\right)\left|S(f;\beta,b,b)\right| \\
= & \left(\frac{\alpha-a}{b-a}\right)\left[\frac{(\alpha-a)^2}{2(\alpha-a)}\left\|f'\right\|_{\infty,[a,\alpha]}\right] \\
& +\left(\frac{\beta-\alpha}{b-a}\right)\left[\frac{(x-\alpha)^2}{2(\beta-\alpha)}\left\|f'\right\|_{\infty,[\alpha,x]}+\frac{(\beta-x)^2}{2(\beta-\alpha)}\left\|f'\right\|_{\infty,[x,\beta]}\right] \\
& +\left(\frac{b-\beta}{b-a}\right)\left[\frac{(b-\beta)^2}{2(b-\beta)}\left\|f'\right\|_{\infty,[\beta,b]}\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(3.10)\qquad & (b-a)\left|\mathfrak{T}(f;a,\alpha,x,\beta,b)\right| \\
\leq & \frac{1}{2}\left\{(\alpha-a)^2\left\|f'\right\|_{\infty,[a,\alpha]}+(x-\alpha)^2\left\|f'\right\|_{\infty,[\alpha,x]}\right. \\
& \left.+(\beta-x)^2\left\|f'\right\|_{\infty,[x,\beta]}+(b-\beta)^2\left\|f'\right\|_{\infty,[\beta,b]}\right\} \\
\leq & \frac{1}{2}\left\{(\alpha-a)^2+(x-\alpha)^2+(\beta-x)^2+(b-\beta)^2\right\}\left\|f'\right\|_{\infty,[a,b]} \\
= & \left\{\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]+\left(\alpha-\frac{a+x}{2}\right)^2+\left(\beta-\frac{x+b}{2}\right)^2\right\} \\
& \times\left\|f'\right\|_{\infty,[a,b]}.
\end{aligned}
$$

The last inequality in (3.10) was obtained in Cerone and Dragomir [13] directly from the three point identity (3.2) for $f(\cdot)$ absolutely continuous. Of course, taking $\alpha=a$ and $\beta=x$ in (3.10) recaptures the Ostrowski inequality as given by the last inequality in (3.9).

Following the procedure established in Section 2 it is possible to obtain a generalised Chebychev identity. From (3.1) and (3.2) we have on multiplying by $\frac{g(x)}{b-a}$ and integrating

$$
\begin{aligned}
(3.11)\qquad & \left(\frac{\beta-\alpha}{b-a}\right)\mathcal{M}(fg;a,b)-\mathcal{M}(f;a,b)\,\mathcal{M}(g;a,b) \\
& +\left[\left(\frac{\alpha-a}{b-a}\right)f(a)+\left(\frac{b-\beta}{b-a}\right)f(b)\right]\mathcal{M}(g;a,b) \\
= & \frac{1}{b-a}\int_a^b\int_a^b g(x)\,r(x,t)\,df(t)\,dx \\
= & \frac{1}{b-a}\left\{\int_a^b g(x)\int_a^x\left(\frac{t-\alpha}{b-a}\right)df(t)\,dx+\int_a^b g(x)\int_x^b\left(\frac{t-\beta}{b-a}\right)df(t)\,dx\right\} \\
= & \frac{1}{b-a}\int_a^b\left[\left(\frac{t-\alpha}{b-a}\right)\int_t^b g(x)\,dx+\left(\frac{t-\beta}{b-a}\right)\int_a^t g(x)\,dx\right]df(t),
\end{aligned}
$$

where we have assumed that an interchange of the order of integration is allowed. Thus, if we define the generalised Chebychev functional as

$$\begin{aligned}(3.12)\qquad &\hat{\mathfrak{T}}(f,g;a,\alpha,x,\beta,b)\\ :&=\left(\frac{\beta-\alpha}{b-a}\right)\mathcal{M}(fg;a,b)-\mathcal{M}(f;a,b)\mathcal{M}(g;a,b)\\ &+\left[\left(\frac{\alpha-a}{b-a}\right)f(a)+\left(\frac{b-\beta}{b-a}\right)f(b)\right]\mathcal{M}(g;a,b)\end{aligned}$$

then from (3.11),

$$\hat{\mathfrak{T}}(f,g;a,\alpha,x,\beta,b)=\frac{1}{(b-a)^2}\int_a^b \chi(t)\,df(t), \tag{3.13}$$

where

$$\chi(t)=(t-\alpha)G(t,b)+(t-\beta)G(a,t). \tag{3.14}$$

It should be emphasised that the earlier functionals may be recaptured by particular instances of the identity (3.13) with (3.12) and (3.14). Specifically,

$$\begin{aligned}\mathfrak{T}(f,g;a,x,b) &= \hat{\mathfrak{T}}(f,g;a,a,x,b,b)\\ \mathcal{M}(g;a,b)\,T(f;a,x,b) &= \hat{\mathfrak{T}}(f,g;a,x,x,x,b)\\ \mathfrak{T}(f;a,x,\beta,b) &= \lim_{h\to 0}\hat{\mathfrak{T}}(f,H(t-x)H(x+h-t);a,\alpha,x,\beta,b)\\ S(f;a,x,b) &= \lim_{h\to 0}\hat{\mathfrak{T}}(f,H(t-x)H(x+h-t);a,a,x,b,b).\end{aligned}$$

It should be remarked that (3.11) and (3.12) may be rewritten in a different form. If we let

$$(b-a)\lambda=\alpha-a,\ \ (b-a)\rho=\beta-\alpha\quad\text{and}\quad(b-a)\nu=b-\beta,$$

so that $\lambda+\rho+\nu=1$, then,

$$\mathfrak{T}(f;a,x,b)=\lambda f(a)+\rho f(x)+\nu f(b)-\mathcal{M}(f;a,b) \tag{3.15}$$

and

$$\hat{\mathfrak{T}}(f,g;a,x,b)=\rho\mathcal{M}(fg;a,b)+[\lambda f(a)+\nu f(b)-\mathcal{M}(f;a,b)]\mathcal{M}(g;a,b).$$

Thus (3.15) is the difference between a weighted combination of a function evaluated at the end points and an interior point x and, the mean of the function over the entire inerval. As a matter of fact,

$$(b-a)\mathfrak{T}(f;a,x,b)=(b-a)[\lambda f(a)+\rho f(x)+\nu f(b)]-\int_a^b f(u)\,du$$

and taking $x=\frac{a+b}{2}$, $\lambda=\nu=\frac{1}{6}$ and $\rho=\frac{2}{3}$ recaptures the Simpson rule under a less restrictive assumption than requiring $f^{(iv)}(\cdot)$ to exist.

4. Ostrowski and Trapezoidal Type Results for $f^{(n)}$, $n \geq 1$

The following Ostrowski and Trapezoidal type identities were obtained, involving $f^{(n)}(\cdot)$, by Cerone et al. [8] and [9]. Namely,

$$(-1)^n \int_a^b P_n(x,t) f^{(n)}(t)\,dt = \int_a^b f(t)\,dt - \sum_{k=1}^{n} \left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] \frac{f^{(k-1)}(x)}{k!}, \tag{4.1}$$

where

$$P_n(x,t) = \begin{cases} \dfrac{(t-a)^n}{n!}, & t \in [a,x] \\[2ex] \dfrac{(t-b)^n}{n!}, & t \in (x,b] \end{cases} \tag{4.2}$$

and

$$(-1)^n \int_a^b Q_n(x,t) f^{(n)}(t)\,dt = \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)}{k!}, \tag{4.3}$$

where

$$Q_n(x,t) = \frac{(t-x)^n}{n!}, \quad t, x \in [a,b]. \tag{4.4}$$

Theorem 6. *The Lebesgue norms for the Ostrowski functional and the generalised functional as given by the right hand sides of (4.1) and (4.2) respectively are equal. That is,*

$$\|P_n(x,\cdot)\|_\gamma = \|Q_n(x,\cdot)\|_\gamma, \quad \gamma \in [1,\infty], \tag{4.5}$$

where $P_n(x,t)$ and $Q_n(x,t)$ are as given by (4.2) and (4.4).

Proof. The proof is straightforward. It may be noticed that over each of the intervals $[a,x]$ and $(x,b]$ the modulus of the two kernels are equivalent. That is,

$$|P_n(x,t)| = \begin{cases} |Q_n(x,a+x-t)|, & t \in [a,x] \\[1ex] |Q_n(x,x+b-t)|, & t \in (x,b]. \end{cases} \tag{4.6}$$

Hence,

$$\begin{aligned} |P_n(x,\cdot)| &= \sup_{t\in[a,b]} |P_n(x,t)| \\ &= \max\left\{ \sup_{t\in[a,x]} |Q_n(x,a+x-t)|, \sup_{t\in(x,b]} |Q_n(x,x+b-t)| \right\} \\ &= \max\left\{ \sup_{t\in[a,x]} |Q_n(x,t)|, \sup_{t\in(x,b]} |Q_n(x,t)| \right\} = \|Q_n(x,\cdot)\|_\infty \end{aligned}$$

and

$$\begin{aligned}
\|P_n(x,\cdot)\|_p^p &= \int_a^b |P_n(x,t)|^p\,dt = \int_a^x |P_n(x,t)|^p\,dt + \int_x^b |P_n(x,t)|^p\,dt \\
&= \int_a^x |Q_n(x,a+x-t)|^p\,dt + \int_x^b |Q_n(x,x+b-t)|^p\,dt \\
&= \int_a^x |Q_n(x,t)|^p\,dt + \int_x^b |Q_n(x,t)|^p\,dt \\
&= \|Q_n(x,\cdot)\|_p^p
\end{aligned}$$

and thus the theorem is proved. ∎

Remark 4. *Cerone et al.* [8] *and* [9] *demonstrated (4.5) by direct calculation. If* $n = 1$ *then the bounds involve* $f^{(1)}(\cdot)$ *and the results of the earlier sections are recaptured for when* $f(\cdot)$ *is absolutely continuous and division by* $-(b-a)$.

Theorem 7. *Let* $\alpha, \beta : [c,d] \to \mathbb{R}$ *be continuous on* $[c,d]$ *and* $|\alpha(t)| = |\beta(c+d-t)|$, *then the Lebesgue norms*

$$\|\alpha\|_\gamma = \|\beta\|_\gamma, \quad \gamma \in [1,\infty]. \tag{4.7}$$

Proof. Firstly,

$$\begin{aligned}
\|\alpha\|_\infty &:= ess \sup_{t\in[c,d]} |\alpha(t)| = ess \sup_{t\in[c,d]} |\beta(c+d-t)| \\
&= ess \sup_{u\in[d,c]} |\beta(u)| = \|\beta\|_\infty .
\end{aligned}$$

Further for $1 \le p < \infty$,

$$\|\alpha\|_p^p := \int_c^d |\alpha(t)|^p\,dt = \int_c^d |\beta(c+d-t)|^p\,dt = \int_c^d |\beta(u)|^p\,du = \|\beta\|_p^p .$$

Thus, the theorem is proved. ∎

Corollary 1. *Let* α_i, $\beta_i : [c_i,d_i] \to \mathbb{R}$ *be continuous on* $[c_i,d_i]$, *where* $I_i = [c_i,d_i] \subset [a,b]$ *and* $\bigcup_{i=1}^n I_i = [a,b]$. *Then if*

$$|\alpha_i(t)| = |\beta_i(c_i+d_i-t)|,$$

the Lebesgue norms on $[a,b]$

$$\|\alpha\|_\gamma = \|\beta\|_\gamma, \quad \gamma \in [0,\infty],$$

where $\alpha,\beta : [a,b] \to \mathbb{R}$ *and* $\alpha(t) = \alpha_i(t)$, $t \in [c_i,d_i]$ *and* $\beta(t) = \beta_i(t)$, $t \in [c_i,d_i]$.

Proof. Follows directly from Theorem 7 applied to each subinterval $[c_i,d_i]$, $i = 1,2,\ldots,n$ with $c_1 = a$ and $c_n = b$. ∎

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