ORDER RELATIONS AND THE OPTIMA OF A CLASS OF REFINED CARLEMAN'S INEQUALITIES

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Abstract. In this paper, we discuss and contrast the weight coefficient of a class of refined Carleman's inequalities obtained in reference [3], and we give their order relations and the best one.

1. Introduction

The Carleman's inequality is the basic and important inequality in mathematics, the basic form of Carleman's inequality is

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{+\infty} a_n, \tag{1.1}$$

where $\{a_n\}_{n=1}^{+\infty}$ is a non-negative sequence such that $0 \le \sum_{n=1}^{+\infty} a_n < +\infty$, and the coefficient e is the best, for details please refer to [1, 2]. Reference[3] has obtained a class of refined Carleman's inequalities, it is

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{+\infty}\left(1-\frac{1-2/e}{n}\right)a_n, \tag{1.2}$$

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\frac{1}{n}\right)^{\frac{1}{\ln 2}-1}}, \tag{1.3}$$

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{+\infty}\frac{\left(1-\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)^{\alpha}}a_n, \tag{1.4}$$

where α, β satisfy $0 \le \alpha \le \frac{1}{\ln 2}-1$, $0 \le \beta \le 1-\frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$.

For our convenience, we write $\alpha_1 = \frac{1}{\ln 2}-1$, $\beta_1 = 1-\frac{2}{e}$, in following.

Inequalities (1.2) and (1.3) are two special cases of (1.4), but we want to know what order relationships among inequalities (1.2), (1.3), and (1.4) there are. In this article, we'll give their order relationships and the best possible one. We have the main theorem:

Theorem 1.1. *Let $\{a_n\}_{n=1}^{+\infty}$ is a non-negative sequence such that $0 \le \sum_{n=1}^{+\infty} a_n < +\infty$, we have inequalities*

$$\sum_{n=1}^{+\infty}(a_1a_2\cdots a_n)^{1/n} \le e\sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\frac{1}{n}\right)^{\alpha_1}} \le e\sum_{n=1}^{+\infty}\frac{\left(1-\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)^{\alpha}}a_n \le e\sum_{n=1}^{+\infty}\left(1-\frac{\beta_1}{n}\right)a_n, \tag{1.5}$$

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where α, β satisfy $0 \le \alpha \le \alpha_1$, $0 \le \beta \le \beta_1$, and $e\beta + 2^{1+\alpha} = e$.

2. The proof of the theorem

In order to prove the theorem, first, we prove a proposition and two lemmas.

Proposition 2.1. *Let*

$$h(x,\alpha,\beta) = \frac{\alpha}{1+x} + \frac{\beta}{1-\beta x} - \frac{\beta_1}{1-\beta_1 x}, \tag{2.1}$$

where α, β *satisfy* $0 \le \alpha \le \alpha_1$, $0 \le \beta \le \beta_1$, *and* $e\beta + 2^{1+\alpha} = e$. *Then we have*

$$h(0,\alpha,\beta) = \alpha + \beta - \beta_1 \ge 0, \tag{2.2}$$

$$h(1,\alpha,\beta) = \frac{\alpha}{2} + \frac{\beta}{1-\beta} - \frac{\beta_1}{1-\beta_1} \le 0. \tag{2.3}$$

Proof. With $\beta = 1 - \frac{2^{1+\alpha}}{e}$, and $\beta_1 = 1 - \frac{2}{e}$, we have

$$h(0,\alpha,\beta) = \frac{2}{e}(1 + \frac{e}{2}\alpha - 2^{\alpha}) \equiv h(0,\alpha),\ \alpha \in [0,\alpha_1], \tag{2.4}$$

$$h(1,\alpha,\beta) = \frac{\alpha}{2} + \frac{e}{2}\frac{1}{2^{\alpha}} - \frac{e}{2} \equiv h(1,\alpha),\ \alpha \in [0,\alpha_1]. \tag{2.5}$$

It is easy to see that

$$\frac{\partial h(0,\alpha)}{\partial \alpha} = \frac{2}{e}(\frac{e}{2} - 2^{\alpha}\ln 2) \ge 1 - \frac{2}{e}2^{\alpha_1}\ln 2 = 1 - \ln 2 > 0, \tag{2.6}$$

$$\frac{\partial h(1,\alpha)}{\partial \alpha} = \frac{1}{2} - \frac{e}{2}\frac{\ln 2}{2^{\alpha}} \le \frac{1}{2} - \frac{e}{2}\frac{\ln 2}{2^{\alpha_1}} = \frac{1}{2} - \ln 2 < 0, \tag{2.7}$$

therefore, $h(0,\ \alpha)$ is monotonic increasing and $h(1,\ \alpha)$ monotone decreasing with $\alpha \in [0,\ \alpha_1]$, respectively. We have

$$h(0,\alpha) = \frac{2}{e}(1 + \frac{e}{2}\alpha - 2^{\alpha}) \ge h(0,0) = 0, \tag{2.8}$$

$$h(1,\alpha) = \frac{\alpha}{2} + \frac{e}{2}\frac{1}{2^{\alpha}} - \frac{e}{2} \le h(1,0) = 0. \tag{2.9}$$

¿From inequalities (2.8) and (2.9), we know that inequalities (2.2) and (2.3) hold. The proof is thus complete. ∎

Lemma 2.2. *For* $m = 1, 2, \cdots$, *inequality*

$$\frac{1 - \frac{\beta}{m}}{(1 + \frac{1}{m})^{\alpha}} \le 1 - \frac{\beta_1}{m} \tag{2.10}$$

holds, where α, β *satisfy* $0 \le \alpha \le \alpha_1$, $0 \le \beta \le \beta_1$, *and* $e\beta + 2^{1+\alpha} = e$.

Proof. Inequality (2.10) is equivalent to

$$1 \le \frac{(1 - \frac{\beta_1}{m})(1 + \frac{1}{m})^{\alpha}}{(1 - \frac{\beta}{m})}. \tag{2.11}$$

Denote

$$f(x,\alpha,\beta) = \frac{(1-\beta_1 x)(1+x)^{\alpha}}{(1-\beta x)} - 1,\ \ x \in (0,1]. \tag{2.12}$$

It is easy to see that

$$\frac{\partial f(x,\alpha,\beta)}{\partial x}=\frac{(1-\beta_1x)(1+x)^{\alpha}}{(1-\beta x)}\left(\frac{\alpha}{1+x}+\frac{\beta}{1-\beta x}-\frac{\beta_1}{1-\beta_1 x}\right). \tag{2.13}$$

It is apparent that $\frac{(1-\beta_1x)(1+x)^{\alpha}}{(1-\beta x)}>0$, $\forall\ x\in(0,\ 1]$.

Denote

$$h(x,\alpha,\beta)=\frac{\alpha}{1+x}+\frac{\beta}{1-\beta x}-\frac{\beta_1}{1-\beta_1 x}. \tag{2.14}$$

It is not difficult to compute that

$$\begin{aligned}\frac{\partial h(x,\alpha,\beta)}{\partial x}&=-\frac{\alpha}{(1+x)^2}+\frac{\beta^2}{(1-\beta x)^2}-\frac{\beta_1^2}{(1-\beta_1x)^2}\\&=-\frac{\alpha}{(1+x)^2}+\frac{(\beta_1-\beta)\left[2\beta\beta_1x-(\beta_1+\beta)\right]}{(1-\beta x)^2(1-\beta_1x)^2}\\&\le-\frac{\alpha}{(1+x)^2}+\frac{(\beta_1-\beta)\left[2\sqrt{\beta\beta_1}-(\beta_1+\beta)\right]}{(1-\beta x)^2(1-\beta_1x)^2},\ x\in(0,1].\end{aligned} \tag{2.15}$$

With $\beta_1-\beta\ge0$, and $2\sqrt{\beta\beta_1}\le\beta_1+\beta$, we have

$$\frac{\partial h(x,\alpha,\beta)}{\partial x}\le-\frac{\alpha}{(1+x)^2}<0,\ x\in(0,1]. \tag{2.16}$$

Inequality (2.16) indicate that $h(x,\ \alpha,\ \beta)$ is monotonic decreasing with $x\in(0,\ 1]$, and according to proposition 2.1, we have

$$\begin{aligned}h(0,\alpha,\beta)&=\alpha+\beta-\beta_1\ge0,\\h(1,\alpha,\beta)&=\frac{\alpha}{2}+\frac{\beta}{1-\beta}-\frac{\beta_1}{1-\beta_1}\le0.\end{aligned} \tag{2.17}$$

Hence, there is only one point x_0 in (0,1] such that $h(x_0,\ \alpha,\ \beta)=0$, namely $\frac{\partial f(x_0,\alpha,\beta)}{\partial x}=0$, and $\frac{\partial f(x,\alpha,\beta)}{\partial x}\ge0$, if $x\in(0,\ x_0)$; $\frac{\partial f(x,\alpha,\beta)}{\partial x}\le0$, if $x\in(x_0,\ 1)$. Therefore, $f(x,\ \alpha,\ \beta)$ is monotonic increasing with $x\in(0,\ x_0)$, and monotonic decreasing with $x\in(x_0,\ 1)$, respectively. Furthermore, $f(0,\ \alpha,\ \beta)=0$, $f(1,\ \alpha,\ \beta)=\frac{(1-\beta_1)2^{\alpha}}{1-\beta}-1=\frac{\frac{2}{e}2^{\alpha}}{\frac{1}{e}2^{1+\alpha}}-1=0$.

We have with the above

$$f(x,\alpha,\beta)\ge\min\{f(0,\alpha,\beta),\ f(1,\alpha,\beta)\}=0,\ x\in(0,1], \tag{2.18}$$

namely

$$\frac{(1-\beta_1x)(1+x)^{\alpha}}{(1-\beta x)}-1\ge0,\quad x\in(0,1]. \tag{2.19}$$

Therefore, inequality (2.11), i.e. (2.10) holds. The proof is thus complete. ∎

Lemma 2.3. *For* $m=1,2,\cdots$, *inequality*

$$\frac{1}{\left(1+\frac{1}{n}\right)^{\alpha_1}}\le\frac{1-\frac{\beta}{m}}{(1+\frac{1}{m})^{\alpha}} \tag{2.20}$$

holds, where α, β *satisfy* $0\le\alpha\le\alpha_1$, $0\le\beta\le\beta_1$, *and* $e\beta+2^{1+\alpha}=e$.

Proof. Inequality (2.20) is equivalent to

$$1 \leq \left(1-\frac{\beta}{m}\right)\left(1+\frac{1}{m}\right)^{\alpha_1-\alpha}. \tag{2.21}$$

Denote

$$g(x,\alpha,\beta)=(1-\beta x)(1+x)^{\alpha_1-\alpha}-1 \quad x\in(0,1]. \tag{2.22}$$

It is easy to see that

$$\frac{\partial g(x,\alpha,\beta)}{\partial x}=(1+x)^{\alpha_1-\alpha-1}\left[(\alpha_1-\alpha-\beta)-\beta(\alpha_1-\alpha+1)x\right]. \tag{2.23}$$

Let

$$u(x,\alpha,\beta)=(\alpha_1-\alpha-\beta)-\beta(\alpha_1-\alpha+1)x. \tag{2.24}$$

Similar to proposition 2.1, it is easy to verify that

$$u(0,\alpha,\beta) = \alpha_1-\alpha-\beta\geq 0, \tag{2.25}$$

$$u(1,\alpha,\beta) = (1-\beta)(\alpha_1-\alpha)-2\beta\leq 0. \tag{2.26}$$

Inequalities (2.25) and (2.26) indicate that there is only one point x_0 in (0, 1] such that $u(x_0,\ \alpha,\ \beta)=0$, namely $\frac{\partial g(x_0,\alpha,\beta)}{\partial x}=0$, and $\frac{\partial g(x,\alpha,\beta)}{\partial x}\geq 0$, if $x\in(0,\ x_0)$; $\frac{\partial g(x,\alpha,\beta)}{\partial x}\leq 0$, if $x\in(x_0,\ 1)$. Therefore $g(x,\ \alpha,\ \beta)$ is monotonic increasing with $x\in(0,\ x_0)$, and monotonic decreasing with $x\in(x_0,\ 1)$, respectively. So we have

$$g(x,\alpha,\beta)\geq\min\{g(0,\alpha,\beta),\ g(1,\alpha,\beta)\}=0, \tag{2.27}$$

and inequality (2.21), i.e. (2.20) holds. The proof of lemma 2.3 is complete. ∎

Finally, with the weight coefficient of inequality (2.10) in lemma 2.2, and (2.20) in lemma 2.3, it is easy to verify that inequalities (1.5) hold. Thus the main theorem 1.1 follows.

Remark 2.1. *According to theorem 1.1, we know that in the weight coefficient form of* $\frac{\left(1-\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)^{\alpha}}$*, where* $\alpha,\ \beta$ *satisfy* $0\leq\alpha\leq\alpha_1,\ 0\leq\beta\leq\beta_1$*, and* $e\beta+2^{1+\alpha}=e$*, the weight coefficient* $\frac{1}{\left(1+\frac{1}{n}\right)^{\alpha_1}}$ *is the best possible, and this is the special case* $\alpha=\alpha_1$*,* $\beta=0$.

References

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