NOTES ON INTEGRAL INEQUALITIES

Nasser Towghi

Raytheon System Company

180 Hartwell Road, Mail Stop S3FG4, Bedford, MA 01730

Nasser_M_Towghi@res.raytheon.com, Phone: 781-999-5546

Abstract

Sufficient conditions for an integral inequality posed as an open question by Feng Qi is given, and proof of previously known integral inequalities under weaker hypothesis are obtained.

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1 Introduction.

In this paper we give sufficient conditions for the following integral inequality to hold,

$$\int_a^b [f(x)]^t \mathrm{d}x \geq \left(\int_a^b f(x) \mathrm{d}x \right)^{t-1} \quad t > 1. \tag{1}$$

This problem was posed by Feng Qi [1]. In [1] it was also shown

$$\int_a^b [f(x)]^{n+2} \mathrm{d}x \geq \left(\int_a^b f(x) \mathrm{d}x \right)^{n+1}, \tag{2}$$

provided that $f^{(n)} \geq n!$, and $f^{(i)}(a) \geq 0$ for $i = 0, .., n-1$.

We give sufficient conditions for inequality (3) to hold. Our proof is somewhat more simplified than the proof of (2) given in [1], and we use slightly weaker assumptions.

We first set the stage. Let $f^{(0)}(x) = f(x)$, $f^{(-1)}(x) = \int_a^x f(s)ds$, and $[x]$ denote the greatest integer less than or equal to x. For $t \in (n, n+1])$, where n is a positive integer, let $\gamma(t) = t(t-1)(t-2)\cdots[t-(n-1)]$. For $t < 1$, let $\gamma(t) = 1$.

Proposition 1 *Let $t > 1$, $x \in [a, b]$, and $f^{(i)}(a) \geq 0$ for $i \leq [t-2]$. If $f^{[t-2]}(x) \geq \gamma(t-1)(x-a)^{(t-[t])}$, then $(b-a)^{t-1} \leq \int_a^b f(x)dx$, and inequality (1) holds.*

Proof. If $1 < t < 2$ then $f^{[t-2]}(b) \leq \int_a^b f(x)dx$, $\gamma(t-1) = 1$, and $(t-[t]) = t-1$. Therefore $(b-a)^{t-1} \leq \int_a^b f(x)dx$. Suppose that $t \in [n, n+1)$, where n is positive integer, and $n \geq 2$. Now

$$\int_a^b f(x)dx \quad \geq \quad \int_a^b \int_a^{x_1} \int_0^{x_2} \cdots \int_a^{x_{n-2}} f^{(n-2)}(x_{n-1}) \mathrm{d}x_{n-1} \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \tag{3}$$

$$
\begin{aligned}
&\geq \int_a^b \int_a^{x_1} \int_a^{x_2} \cdots \int_a^{x_{n-2}} \gamma(t-1)(x_{n-1}-a)^{(t-n)} \mathrm{d}x_{n-1} \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \\
&= (b-a)^{(t-1)}.
\end{aligned}
$$

To show that inequality (1) holds we define the following quantities,

$$
F(t) = \int_a^b [f(x)]^t \mathrm{d}x, \quad G(t) = \left[\int_a^b f(x)\mathrm{d}x\right]^t, \quad G_I(t) \quad = \quad \left[\frac{G(1)}{b-a}\right]^t. \tag{4}
$$

We must show that $G(t-1) \leq F(t)$. Note that $G(t-1) = G_I(t)\frac{(b-a)^t}{G(1)}$. By Jenson's inequality $G_I(t) \leq F(t)/(b-a)$. Consequently

$$
\begin{aligned}
G(t-1) &\leq F(t)\frac{(b-a)^{t-1}}{G(1)} && (5) \\
(\text{Inequality (3)} \Rightarrow) &\leq F(t).
\end{aligned}
$$

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References

[1] Feng Qi, “SEVERAL INTEGRAL INEQUALITIES,” Journ. of Inequalities in Pure and Applied Math. Vol.1 no. 2 Article 19 (2000).