INEQUALITIES FOR MATHIEU'S SERIES

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ABSTRACT. In the article, using the integral expression of Mathieu's series and by some integral inequalities involving periodic functions, several new inequalities and estimates for the Mathieu's series are presented.

CONTENTS

1. INTRODUCTION

In 1890, Mathieu defined in [8] $S(r)$ as

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2},\quad r>0, \tag{1}$$

and conjectured that $S(r)<1/r^2$. We call formula (1) Mathieu's series.

In [7], Makai proved

$$\frac{1}{r^2+1/2}<S(r)<\frac{1}{r^2}. \tag{2}$$

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H. Alzer, J. L. Brenner and O. G. Ruehr in [1] obtained

$$\frac{1}{x^2+1/(2\zeta(3))} < S(x) < \frac{1}{x^2+1/6}, \tag{3}$$

where ζ denotes the zeta function.

The integral form of Mathieu's series (1) was given in [3, 4] by

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x}{\mathrm{e}^x - 1}\sin(rx)\,\mathrm{d}x. \tag{4}$$

There has been a rich literature on the study of Mathieu's series and its inequalities, we can find many more interesting refinements and extensions of Mathieu's inequality in [1]–[11].

In this paper, using the integral expression (4) of Mathieu's series and certain inequalities involving periodic function, some new inequalities for Mathieu's series are established. At last, three open problems are proposed.

2. Lemmae

Lemma 2.1. *Let $\psi(x)$ be an integrable function satisfying $\psi(x) = -\psi(x+T)$, where T is a given positive number, and $\psi(x) \geqslant 0$ for $x \in [0,T]$, let $f(x)$ and $g(x)$ be two integrable functions on $[0,2T]$ such that*

$$f(x) - g(x) \geqslant f(x+T) - g(x+T) \tag{5}$$

on $[0,T]$. Then

$$\int_0^{2T} \psi(x) f(x)\,\mathrm{d}x \geqslant \int_0^{2T} \psi(x) g(x)\,\mathrm{d}x. \tag{6}$$

Proof. By easy computation, it is deduced that

$$\begin{aligned}
&\int_0^{2T} \psi(x)[f(x)-g(x)]\,\mathrm{d}x \\
&= \int_0^{T} \psi(x)[f(x)-g(x)]\,\mathrm{d}x + \int_T^{2T} \psi(x)[f(x)-g(x)]\,\mathrm{d}x \\
&= \int_0^{T} \psi(x)[f(x)-g(x)]\,\mathrm{d}x + \int_0^{T} \psi(x+T)[f(x+T)-g(x+T)]\,\mathrm{d}x \\
&= \int_0^{T} \psi(x)\big\{[f(x)-g(x)] - [f(x+T)-g(x+T)]\big\}\,\mathrm{d}x \\
&\geqslant 0.
\end{aligned}$$

The proof is complete. ∎

Corollary 2.1. *Let $\psi(x) \not\equiv 0$ be an integrable periodic function with period $2T > 0$ satisfying $\psi(x) = -\psi(x+T)$ and $\psi(x) \geqslant 0$ for $x \in [0,T]$. If $f(x)$ is an integrable function such that $f(x) \geqslant f(x+T)$ on $[0,T]$, then*

$$\int_0^{2T} \psi(x) f(x)\,\mathrm{d}x \geqslant 0. \tag{7}$$

Corollary 2.2. *Let $f(x)$ be an integrable function such that $f(x) \geqslant f(x+\pi)$ on $[0,\pi]$, then*

$$\int_0^{2\pi} f(x)\sin x\,\mathrm{d}x \geqslant 0. \tag{8}$$

3. Main Results

As a direct consequence of Lemma 2.1, we have

Theorem 3.1. *Let Φ_1 and Φ_2 be two integral functions sush that $\frac{x}{\mathrm{e}^x-1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{x}{\mathrm{e}^x-1}$ are increasing respectively. Then, for any positive number r, we have*

$$\frac{1}{r}\int_0^{\infty} \Phi_2(x)\sin(rx)\,\mathrm{d}x \leqslant \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \leqslant \frac{1}{r}\int_0^{\infty} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{9}$$

Proof. The function $\psi(x) = \sin(rx)$ has a period $\frac{2\pi}{r}$, and $\psi(x) = -\psi\left(x + \frac{\pi}{r}\right)$.

Since $f(x) = \frac{x}{\mathrm{e}^x-1} - \Phi_1(x)$ is increasing, for any $\alpha > 0$, we have $f(x+\alpha) \geqslant f(x)$. Therefore, from Corollary 2.1, we obtain

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \left[\frac{x}{\mathrm{e}^x-1} - \Phi_1(x)\right]\sin(rx)\,\mathrm{d}x \leqslant 0, \tag{10}$$

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x}{\mathrm{e}^x-1}\sin(rx)\,\mathrm{d}x \leqslant \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{11}$$

Then, from formula (4), we have

$$\begin{aligned} S(r) &= \frac{1}{r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x}{\mathrm{e}^x-1}\sin(rx)\,\mathrm{d}x \\ &\leqslant \frac{1}{r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x)\sin(rx)\,\mathrm{d}x \\ &= \frac{1}{r}\int_0^{\infty} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \end{aligned} \tag{12}$$

The right hand side of inequality (9) follows.

Similar arguments yield the left hand side of inequality (9). ∎

Proposition 3.1. *The function*

$$g(x) = \frac{x}{e^x - 1} - \frac{x^2}{e^{3x} - e^x} \tag{13}$$

is decreasing with $x > 0$.

Proof. By straightforward computation, we have

$$\begin{aligned} g'(x) &= -\frac{e^{-x}(e^x + e^{2x} - e^{3x} - e^{4x} - 2x + 3x e^{2x} + 2x e^{3x} + x e^{4x} - 3x^2 e^{2x} + x^2)}{(e^{2x} - 1)^2} \\ &\equiv -\frac{h(x) e^{-x}}{(e^{2x} - 1)^2}, \end{aligned} \tag{14}$$

$$h'(x) = -2 + e^x + 5 e^{2x} - e^{3x} - 3 e^{4x} + 6x e^{3x} + 4x e^{4x} - 6x^2 e^{2x}, \tag{15}$$

$$h''(x) = 2 + e^x + 10 e^{2x} + 3 e^{3x} - 8 e^{4x} - 12x e^{2x} + 18x e^{3x} + 16x^2 e^{4x} - 12x^2 e^{2x}, \tag{16}$$

$$h'''(x) = e^x(1 + 8 e^x + 27 e^{2x} - 16 e^{3x} - 48x e^x + 54x e^{2x} + 64x e^{3x} - 24x^2 e^x), \tag{17}$$

$$\left(\frac{h'''(x)}{e^x}\right)' = 4 e^x(-10 + 27 e^x + 4 e^{2x} - 24x + 27x e^x + 48x e^{2x} - 6x^2), \tag{18}$$

$$\left(\frac{(h'''(x)/e^x)'}{4 e^x}\right)' = (54 e^x - 24) + 56 e^{2x} + (27x e^x - 12x) + 96x e^{2x} > 0. \tag{19}$$

Therefore, the function $(h'''(x)/e^x)'/4 e^x$ is increasing. Since $[(h'''(x)/e^x)'/4 e^x]|_{x=0} = 21$, then $(h'''(x)/e^x)' > 0$, and $h'''(x)/e^x$ is increasing, $[h'''(x)/e^x]|_{x=0} = 20$, $h'''(x) > 0$, $h''(x)$ increases, $h''(0) = 8$, $h''(x) > 0$, $h'(x)$ increases, $h'(0) = 0$, $h'(x) \geqslant 0$, $h(x)$ increases, $h(0) = 0$, then $h(x) \geqslant 0$, $g'(x) \leqslant 0$ for $x > 0$, the function $g(x)$ is decreasing. The proof follows. ∎

Corollary 3.1. *For any positive number $r > 0$, we have*

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \geqslant \frac{1}{8r(1 + r^2)^3} \left[16r(r^2 - 3) + \pi^3(r^2 + 1)^3 \operatorname{sech}^2\left(\frac{\pi r}{2}\right) \tanh\left(\frac{\pi r}{2}\right)\right]. \tag{20}$$

Proof. In [12, p. 356], the following formula is given

$$\int_0^\infty \frac{x^{2m}\sin(ax)}{\mathrm{e}^{(2n+1)\alpha x}-\mathrm{e}^{(2n-1)\alpha x}}\,\mathrm{d}x = (-1)^m \frac{\partial^{2m}}{\partial a^{2m}}\left[\frac{\pi}{4\alpha}\tanh\frac{a\pi}{2\alpha} - \sum_{\nu=1}^{n}\frac{a}{a^2+(2\nu-1)^2\alpha^2}\right], \tag{21}$$

where $\alpha > 0$ and $m, n = 0, 1, 2, \dots$. If $n = 0$ in formula (21), then the summation terms are omitted.

Therefore, we have

$$\begin{aligned}\int_0^\infty \frac{x^2\sin(rx)}{\mathrm{e}^{3x}-\mathrm{e}^{x}}\,\mathrm{d}x &= -\frac{\partial^2}{\partial r^2}\left[\frac{\pi}{4}\tanh\frac{\pi r}{2} - \frac{r}{r^2+1}\right]\\ &= \frac{1}{8(1+r^2)^3}\left[16r(r^2-3)+\pi^3(r^2+1)^3\operatorname{sech}^2\left(\frac{\pi r}{2}\right)\tanh\left(\frac{\pi r}{2}\right)\right].\end{aligned} \tag{22}$$

From Theorem 3.1 and Proposition 3.1, inequality (20) follows. ∎

Remark 3.1. By numerical calculation, we can find that the lower bound of the inequality (20) is better than that of (3) if $0.5625 < r < 1.784$.

Theorem 3.2. *Suppose c is a positive number, then, for any positive real number α, we have*

$$\frac{1}{c^2+\frac{1}{2}} < \sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^\alpha+c^2)^2} < \frac{1}{c^2}. \tag{23}$$

Proof. By standard argument, we obtain

$$\begin{aligned}&\frac{1}{(n^{\alpha/2}-\frac{1}{2})^2+c^2-\frac{1}{4}} - \frac{1}{(n^{\alpha/2}+\frac{1}{2})^2+c^2-\frac{1}{4}}\\ =\ &\frac{2n^{\alpha/2}}{(n^\alpha+c^2-n^{\alpha/2})(n^\alpha+c^2+n^{\alpha/2})}\\ >\ &\frac{2n^{\alpha/2}}{(n^\alpha+c^2)^2}\\ >\ &\frac{2n^{\alpha/2}}{(n^\alpha+c^2)^2+c^2+\frac{1}{4}}\\ =\ &\frac{1}{(n^{\alpha/2}-\frac{1}{2})^2+c^2+\frac{1}{4}} - \frac{1}{(n^{\alpha/2}+\frac{1}{2})^2+c^2+\frac{1}{4}},\end{aligned}$$

the proof is complete. ∎

4. Open Problems

In [5], Professor B.-N. Guo proposed the following

Open Problem 4.1. *Let*

$$S(r,t)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^{t+1}},\tag{24}$$

where $t>0$ and $r>0$. Can one obtain an integral representation of $S(r,t)$ similar to (4)*?*

Now we propose two similar open problems below

Open Problem 4.2. *Let*

$$S(r,\alpha)=\sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^\alpha+r^2)^2},\tag{25}$$

where $r>0$ and $\alpha>0$. Can one establish an integral expression of $S(r,\alpha)$?

Open Problem 4.3. *Let*

$$S(r,t,\alpha)=\sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^\alpha+r^2)^{t+1}}\tag{26}$$

for $t>0$, $r>0$ and $\alpha>0$. Can one obtain an integral expression of $S(r,t,\alpha)$?

In [12, p. 356], the following formula is given

$$\int_0^\infty\frac{x^{2m}\sin(ax)}{\mathrm{e}^{2n\alpha x}-\mathrm{e}^{(2n-2)\alpha x}}\,\mathrm{d}x=(-1)^m\frac{\partial^{2m}}{\partial a^{2m}}\left[\frac{\pi}{4\alpha}\coth\frac{a\pi}{2\alpha}-\frac{1}{2a}-\sum_{\nu=1}^{n-1}\frac{a}{a^2+(2\nu)^2\alpha^2}\right],\tag{27}$$

where $a>0$, $\alpha>0$, $m=0,1,2,\dots$ and $n=1,2,\dots$. If $n=1$ in formula (27), then the summation terms are omitted.

Open Problem 4.4. *Find suitable ranges of numbers a, α, m and n such that*

$$\frac{x}{\mathrm{e}^x-1}-\frac{x^{2m}}{\mathrm{e}^{(2n+1)\alpha x}-\mathrm{e}^{(2n-1)\alpha x}},\quad \alpha>0\quad\text{and}\quad m,n=0,1,2,\dots\tag{28}$$

or

$$\frac{x}{\mathrm{e}^x-1}-\frac{x^{2m}}{\mathrm{e}^{2n\alpha x}-\mathrm{e}^{(2n-2)\alpha x}},\qquad \alpha>0,\quad m=0,1,2,\dots\quad\text{and}\quad n=1,2,\dots\tag{29}$$

are monotonic in x respectively.

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