AN INEQUALITY FOR THE RATIOS OF THE ARITHMETIC MEANS OF FUNCTIONS WITH A POSITIVE PARAMETER

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Abstract. In the article, an integral inequality for the ratios of the arithmetic means of functions with a positive parameter are obtained, and an open problem, posed by B.-N. Guo and F. Qi in "*An algebraic inequality, II*, RGMIA Research Report Collection **4** (2001), no. 1, Article 8 (Available online at `http://rgmia.vu.edu.au/v4n1.html`)", is resolved partially.

Contents

1. Introduction

In the papers [2, 3], using the Cauchy's mean-value theorem and an inequality between the logarithmic mean and one-parameter mean, Dr. F. Qi and Professor B.-N. Guo proved that, if $b > a > 0$ and $\delta > 0$ be real numbers, then, for any given

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positive number $r > 0$, we have

$$\frac{b}{b+\delta} < \left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} = \left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1}$$

The lower and upper bounds in (1) are the best possible.

Note that, in [2], a rich literature related to inequality (1) and its history and background are listed.

Meanwhile, they posed an open problem in [2] as follows: Let $b > a > 0$ and $\delta > 0$ be real numbers, $f(x)$ a positive integrable function, then, for any given positive parameter $r > 0$, we have

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)} < \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} f^r(x)\,\mathrm{d}x}\right)^{1/r} < \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,\mathrm{d}x - \frac{1}{b+\delta-a}\int_a^{b+\delta} \ln f(x)\,\mathrm{d}x\right). \tag{2}$$

The lower and upper bounds in (2) are the best possible.

It is well-known that the arithmetic mean of function $f(t)$ on the closed interval $[r,s]$ is defined as

$$\phi(r,s) = \begin{cases} \frac{1}{s-r}\int_r^s f(t)\,\mathrm{d}t, & r \neq s; \\ f(r), & r = s. \end{cases} \tag{3}$$

In this paper, we will resolve the above conjecture partially and obtain the following

Theorem 1. *Let $f(x) \not\equiv 0$ be a nonnegative integrable function on the closed interval $[a, b+\delta]$, where $b > a$ and $\delta > 0$. Then, for any positive parameter $r > 0$, we have*

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)} \le \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} f^r(x)\,\mathrm{d}x}\right)^{1/r}. \tag{4}$$

Theorem 2. *There exists a positive function $f(x)$ defined on the closed interval $[0,3]$ such that $f^r(x)$ and $\ln f(x)$ being integrable on $[0,3]$, and*

$$\left(\frac{\frac{1}{2}\int_0^2 f^r(x)\,\mathrm{d}x}{\frac{1}{3}\int_0^3 f^r(x)\,\mathrm{d}x}\right)^{1/r} > \exp\left(\frac{\int_0^2 \ln f(x)\,\mathrm{d}x}{2} - \frac{\int_0^3 \ln f(x)\,\mathrm{d}x}{3}\right), \quad r > 0. \tag{5}$$

Remark 1. It is natural to ask that, what conditions does the function f satisfy, the right hand side of inequality (2) holds? If f is continous, monotonic, or convex, does it hold?

2. Lemma

To prove Theorem 1, the following lemma is necessary.

Lemma 1. *Let $r > 0$ be a positive real number, let a_i, $1 \le i \le n$, be a nonnegative sequence and $\infty > \alpha \ge \max_{1\le i\le n}\{a_i\} > 0$ a constant. Define*

$$F_n(r) = \frac{\sum_{i=1}^n a_i^r}{\sum_{i=1}^n a_i^r + n\alpha^r}, \quad r > 0. \tag{6}$$

Then $0 \le F_n(r) \le \frac{1}{2}$, and the functions $F_n(r)$, $[F_n(r)]^{1/r}$ and $[2F_n(r)]^{1/r}$ are decreasing.

Proof. It is trivial to see that $0 \le F_n(r) \le \frac{1}{2}$.

Direct differentiation gives us

$$\frac{\mathrm{d}F_n(r)}{\mathrm{d}r} = \frac{n\alpha^r \sum_{i=1}^n a_i^r \ln\left(\frac{a_i}{\alpha}\right)}{\left(\sum_{i=1}^n a_i^r + n\alpha^r\right)^2} < 0,$$

therefore, $F_n(r)$ is a decreasing function of r.

The function $a^{1/t}$ is a strictly increasing function of t on the closed interval $[0, 1]$ for $0 < a < 1$. Let $r < s$, then

$$[F_n(r)]^{1/r} \ge [F_n(s)]^{1/r} > [F_n(s)]^{1/s},$$

$$[2F_n(r)]^{1/r} \ge [2F_n(s)]^{1/r} \ge [2F_n(s)]^{1/s}.$$

Thus, the functions $[F_n(r)]^{1/r}$ and $[2F_n(r)]^{1/r}$ are decreasing. □

3. Proofs of Theorems

In this section, we will prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Assume that f is integrable in the sense of Riemann. Taking a partition $P_1 = (x_0, x_1, \dots, x_n)$ of the closed interval $[a, b]$ with $x_i = a + \frac{i(b-a)}{n}$ for $0 \le i \le n$ and a partition $P_2 = (x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_{2n})$ of the closed interval $[a, b+\delta]$ with $x_j = b + \frac{(j-n)\delta}{n}$ for $n+1 \le j \le 2n$, by definition of Riemann integral (see [1]), we have

$$\int_a^b f^r(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n f^r(x_i), \tag{7}$$

$$\int_a^{b+\delta} f^r(x)\,\mathrm{d}x = \lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{i=1}^{n} f^r(x_i) + \frac{\delta}{n}\sum_{j=n+1}^{2n} f^r(x_j)\right). \tag{8}$$

Let $\alpha = \sup_{x\in[a,b+\delta]} f(x)$ and $a_i = f(x_i)$ for $1 \le i \le n$. From formulae (7) and (8) and using the notations of Lemma 1, it follows that

$$\begin{aligned}
&\left(\frac{\int_a^b f^r(x)\,\mathrm{d}x}{b-a}\Big/\frac{\int_a^{b+\delta} f^r(x)\,\mathrm{d}x}{b+\delta-a}\right)^{1/r}\\
&=\left(\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n f^r(x_i)}{\lim_{n\to\infty}\left[\frac{b-a}{n(b+\delta-a)}\sum_{i=1}^n f^r(x_i)+\frac{\delta}{n(b+\delta-a)}\sum_{j=n+1}^{2n} f^r(x_j)\right]}\right)^{1/r}\\
&=\left(\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{i=1}^n f^r(x_i)}{\frac{b-a}{n(b+\delta-a)}\sum_{i=1}^n f^r(x_i)+\frac{\delta}{n(b+\delta-a)}\sum_{j=n+1}^{2n} f^r(x_j)}\right)^{1/r}\\
&=\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n f^r(x_i)}{\frac{b-a}{b+\delta-a}\sum_{i=1}^n f^r(x_i)+\frac{\delta}{b+\delta-a}\sum_{j=n+1}^{2n} f^r(x_j)}\right)^{1/r}\\
&\ge\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n f^r(x_i)}{\sum_{i=1}^n f^r(x_i)+\sum_{j=n+1}^{2n} f^r(x_j)}\right)^{1/r}\\
&\ge\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n f^r(x_i)}{\sum_{i=1}^n f^r(x_i)+n\alpha^r}\right)^{1/r}\\
&=\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n a_i^r}{\sum_{i=1}^n a_i^r+n\alpha^r}\right)^{1/r}\\
&=\lim_{n\to\infty}\left(\frac{\sum_{i=1}^n a_i^r}{\sum_{i=1}^n a_i^r+n\alpha^r}\right)^{1/r}\\
&=\lim_{n\to\infty}[F_n(r)]^{1/r} \qquad \text{(by definition of } F_n(r))\\
&\ge\lim_{n\to\infty}\lim_{r\to\infty}[F_n(r)]^{1/r} \qquad \text{(since } [F_n(r)]^{1/r} \text{ is strictly decreasing)}\\
&=\lim_{n\to\infty}\frac{\max_{1\le i\le n}\{a_i\}}{\alpha} \qquad \text{(by the L'Hospital rule)}\\
&=\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)}.
\end{aligned}$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Define

$$f(x)=\begin{cases}\varepsilon, & x\in[0,1);\\ 1, & x\in[1,2);\\ \varepsilon^\beta, & x\in[2,3];\end{cases} \tag{9}$$

where $\varepsilon > 0$ and β is a given constant. A calculation shows that

$$\left(\frac{\frac{1}{2}\int_0^2 f^r(x)\,\mathrm{d}x}{\frac{1}{3}\int_0^3 f^r(x)\,\mathrm{d}x}\right)^{1/r} = \left(\frac{3(1+\varepsilon^r)}{2[1+\varepsilon^r+\varepsilon^{\beta r}]}\right)^{1/r},$$

$$\exp\left(\frac{\int_0^2 \ln f(x)\,\mathrm{d}x}{2} - \frac{\int_0^3 \ln f(x)\,\mathrm{d}x}{3}\right) = \varepsilon^{\frac{1-2\beta}{6}},$$

$$\frac{\mathrm{d}h_\varepsilon(r)}{\mathrm{d}r} \triangleq \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1+\varepsilon^r}{1+\varepsilon^r+\varepsilon^{\beta r}}\right) = \frac{\varepsilon^{(1+\beta)r}\ln\varepsilon}{[1+\varepsilon^r+\varepsilon^{\beta r}]^2} \triangleq g_\varepsilon(r).$$

If $0 < \varepsilon < 1$, the function $g_\varepsilon(r) < 0$, and $h_\varepsilon(r)$ is decreasing with $r > 0$, then $h_\varepsilon(r) > \lim_{r\to\infty} h_\varepsilon(r) = 1$. If $\varepsilon > 1$, the function $g_\varepsilon(r) > 0$, and $h_\varepsilon(r)$ is increasing with $r > 0$, then $h_\varepsilon(r) > \lim_{r\to 0} h_\varepsilon(r) = \frac{2}{3}$. Therefore, for $0 < \varepsilon < 1$ and $\beta < \frac{1}{2}$, or for $\varepsilon > 1$ and $\beta > \frac{1}{2}$, we have

$$\left(\frac{3(1+\varepsilon^r)}{2[1+\varepsilon^r+\varepsilon^{\beta r}]}\right)^{1/r} > 1 > \varepsilon^{\frac{1-2\beta}{6}}, \quad r > 0.$$

The proof of Theorem 2 is complete. □

References

[1] R. Beals, *Advanced Mathematical Analysis*, Graduate Texts in Mathematics **12**, Springer-Verlag, New York·Heidelberg·Berlin, 1973. Reprinted in China by World Publishing Corporation, Beijing, China.

[2] B.-N. Guo and F. Qi, *An algebraic inequality, II*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Article 8. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

[3] F. Qi, *An algebraic inequality*, J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Article 13. Available online at `http://jipam.vu.edu.au/v2n1/006_00.html`. RGMIA Res. Rep. Coll. **2** (1999), no. 1, Article 8, 81–83. Available online at `http://rgmia.vu.edu.au/v2n1.html`.

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