APPROXIMATE MULTIDIMENSIONAL INTEGRATION THROUGH DIMENSION REDUCTION VIA THE OSTROWSKI FUNCTIONAL

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Abstract. An iterative approach is used to represent multidimensional integrals in terms of lower dimensional integrals and function evaluations. The procedure is quite general utilising one dimensional identities as the *seed* or *generator* to procure multidimensional identities. Bounds are obtained from the identities.

1. Introduction

We firstly review the Ostrowski type results obtained involving one dimensional integrals.

For $f:[a,b]\to\mathbb{R}$ we define the Ostrowski functional by

$$S(f;c,x,d):=f(x)-\mathcal{M}(f;c,d), \tag{1.1}$$

where

$$\mathcal{M}(f;c,d):=\frac{1}{d-c}\int_c^d f(u)\,du, \text{ the integral mean.} \tag{1.2}$$

We note that

$$(b-a)\,S\left(f;a,\frac{a+b}{2},b\right)=(b-a)\,f\left(\frac{a+b}{2}\right)-\int_a^b f(u)\,du, \tag{1.3}$$

recapturing the midpoint rule for the evaluation of the integrals. With this in mind, the most common task is to obtain bounds on the above functionals. This task is perhaps best accomplished from an identity involving the functionals. The following identity may be easily shown to hold, for f of bounded variation, by an integration by parts argument of the Riemann-Stieltjes integrals and so

$$S(f;c,x,d)=\int_c^d p(x,t,c,d)\,df(t), \quad p(x,t,c,d)=\begin{cases}\dfrac{t-c}{d-c}, & t\in[c,x]\\[2ex] \dfrac{t-d}{d-c}, & t\in(x,d].\end{cases} \tag{1.4}$$

Further, if $f(t)$ is assumed to be absolutely continuous for t over its respective interval, then $df(t)=f'(t)\,dt$ and the Riemann-Stieltjes integrals in (1.4) is equivalent to a Riemann integral. In this instance the corresponding identity to (1.4) is known as Montgomery's identity (see [6]).

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The following theorem may be proved using the Montgomery identity (see Fink [22] and Dragomir and Wang [19] – [21]).

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$, we have:*

$$|S(f;a,x,b)| \tag{1.5}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right](b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f' \in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases} \tag{1.6}$$

where $S(f;a,x,b)$ is as given by (1.1), $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesque norms on $L_r[a,b]$, namely,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}}, \qquad r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense that they cannot be replaced by a smaller constant.

Ostrowski [26] proved the first inequality in (1.5) in 1938, using a different argument and hence the justification for the naming of $S(f;a,x,b)$. Fink [22] also obtained generalisations of the above results as did Anastassiou [1]. See also Dragomir and Rassias [18], a book devoted to Ostrowski inequalities.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [17]):

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be of $r - H-$Hölder type, that is,*

$$|f(x) - f(y)| \leq H|x-y|^r, \qquad \textit{for all } x,y \in [a,b],$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then, for all $x \in [a,b]$, we have the inequality:

$$|S(f;a,x,b)| \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r, \tag{1.7}$$

where $S(f;a,x,b)$ is as given by (1.1).
The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, that is f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H)

(see [13])

$$|S(f;a,x,b)| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.8}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [11]).

Theorem 3. *Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then*

$$|S(f;a,x,b)| \leq \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.9}$$

for all $x\in[a,b]$.
The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, that is, f is monotonically increasing, then the inequality (1.9) may be improved in the following manner [12].

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be monotonic nondecreasing. Then for all $x\in[a,b]$, we have the inequalities:*

$$\begin{aligned} &|S(f;a,x,b)| \\ \leq\ & \frac{1}{b-a}\left\{[2x-(a+b)]f(x)+\int_a^b sgn(t-x)f(t)\,dt\right\} \\ \leq\ & \frac{1}{b-a}\left\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\right\} \\ \leq\ & \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)], \end{aligned} \tag{1.10}$$

where $S(f;a,x,b)$ is as given by (1.1).
All the inequalities in (1.10) are sharp and the constant $\frac{1}{2}$ is the best possible.

The interested reader is encouraged to see [8] for an extensive treatment of the Ostrowski (interior point) rule and its applications to numerical quadrature. Also, for other recent results including Ostrowski type inequalities for $n-$time differentiable functions, visit the RGMIA website at `http://rgmia.vu.edu.au/database.html`.

It is the aim of the current paper to extend the above results to multidimensional integrals and to provide explicit bounds. We now outline some of the existing results in this area.

In 1975, G.N. Milovanović generalized the first inequality in Theorem 1 due to Ostrowski [26], to the case where f is a function of several variables.

Following [24], let $D=\{(x_1,\ldots,x_m)\,|a_i<x_i<b_i\ \ (i=1,\ldots,m)\}$ and let $\bar{D}$ be the closure of D, then we have the following generalisation of Theorem 1.

Theorem 5. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial x_1}\right|\leq M_i$ $(M_i>0;\ i=1,\dots,m)$ in D. Then, for every $X=(x_1,\dots,x_m)\in\bar{D}$,*

$$\left|f(X)-\frac{1}{\prod_{i=1}^{m}(b_i-a_i)}\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}f(y_1,\dots,y_m)\,dy_1\cdots dy_m\right| \tag{1.11}$$
$$\leq \sum_{i=1}^{m}\left[\frac{1}{4}+\frac{\left(x_i-\frac{a_i+b_i}{2}\right)^2}{(b_i-a_i)^2}\right](b_i-a_i)\,M_i.$$

The following weighted version can also be found in [24].

Theorem 6. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be a differentiable function defined on*

$$D=\{(x_1,...,x_m)\,|a_i\leq x_i\leq b_i\,(i=1,...,m)\}$$

and let $\left|\frac{\partial f}{\partial x_i}\right|\leq M_i$ $(M_i>0,i=1,...,m)$ in D. Furthermore, let function $x\longmapsto p(x)$ be integrable and $p(x)>0$ for every $x\in D$. Then for every $x\in D$, we have the inequality:

$$\left|f(x)-\frac{\int_D p(y)\,f(y)\,dy}{\int_D p(y)\,dy}\right|\leq\frac{\sum_{i=1}^{m}M_i\int_D p(y)\,|x_i-y_i|\,dy}{\int_D p(y)\,dy}. \tag{1.12}$$

The following result was obtained in [15] for $f(\cdot)$ Hölder continuous.

Theorem 7. *Assume that the mapping $f:[a_1,b_1]\times...\times[a_n,b_n]\to\mathbb{R}$ satisfies the following $r-$Hölder type condition:*

$$|f(\bar{\mathbf{x}})-f(\bar{\mathbf{y}})|\leq\sum_{i=1}^{n}L_i\,|x_i-y_i|^{r_i}\,(L_i\geq 0,i=1,...,n) \tag{H}$$

for all $\bar{\mathbf{x}}=(x_1,...,x_n)$, $\bar{\mathbf{y}}=(y_1,...,y_n)\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]:=[a_1,b_1]\times...\times[a_n,b_n]$, where $r_i\in(0,1]$, $i=1,...,n$. We have then the Ostrowski type inequality:

$$\left|f(\bar{\mathbf{x}})-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f(\bar{\mathbf{t}})\,d\bar{\mathbf{t}}\right| \tag{1.13}$$
$$\leq \sum_{i=1}^{n}\frac{L_i}{r_i+1}\left[\left(\frac{x_i-a_i}{b_i-a_i}\right)^{r_i+1}+\left(\frac{b_i-x_i}{b_i-a_i}\right)^{r_i+1}\right](b_i-a_i)^{r_i}$$
$$\leq \sum_{i=1}^{n}\frac{L_i\,(b_i-a_i)^{r_i}}{r_i+1}$$

for all $\bar{\mathbf{x}}\in\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$, where $\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}}f(\bar{\mathbf{t}})\,d\bar{\mathbf{t}}=\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(t_1,...,t_n)\,dt_n...dt_1$.

The following generalisation of Theorem 7 holds (see [15]) which involves a weighted multidimensional Ostrowski type result for $f(\cdot)$ Hölder continuous.

Theorem 8. *Let $f,w:\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]\to\mathbb{R}$ such that f is of $r-$Hölder type with the constants L_i and $r_i\in(0,1]$ $(i=1,\dots,n)$ and where w is integrable on $\left[\bar{\mathbf{a}},\bar{\mathbf{b}}\right]$,* *nonnegative on this interval and*

$$\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{x}})\, d\bar{\mathbf{x}} := \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} w(x_1, \dots, x_n)\, dx_n \dots dx_1 > 0.$$

We then have the inequality,

$$\left| f(\bar{\mathbf{x}}) - \frac{1}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}} \int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}}) f(\bar{\mathbf{y}})\, d\bar{\mathbf{y}} \right| \le \sum_{i=1}^{n} L_i \frac{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} |x_i - y_i|^{r_i} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}}{\int_{\bar{\mathbf{a}}}^{\bar{\mathbf{b}}} w(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}}, \tag{1.14}$$

for all $\bar{\mathbf{x}} \in \left[\bar{\mathbf{a}}, \bar{\mathbf{b}}\right]$.

The following theorem was proved in [3] by examining the four regions of the $s-t$ plane.

Theorem 9. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be so that* $f(\cdot,\cdot)$ *is integrable on* $[a,b] \times [c,d]$, $f(x,\cdot)$ *is integrable on* $[c,d]$ *for any* $x \in [a,b]$ *and* $f(\cdot,y)$ *is integrable on* $[a,b]$ *for any* $y \in [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$ *and is bounded, i.e.,*

$$\left\| f''_{s,t} \right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty$$

then we have the inequality:

$$\begin{aligned} &\left| \int_a^b \int_c^d f(s,t)\, ds dt - \left[(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(s,y)\, ds \right.\right. \\ &\left.\left. - (d-c)(b-a) f(x,y) \right] \right| \\ \le\; & \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty \end{aligned} \tag{1.15}$$

for all $(x,y) \in [a,b] \times [c,d]$.

It should be noted that the tightest bound is obtained if x and y are taken at their respective midpoints and the constants $\frac{1}{4}$ are best possible in the sense that they cannot be replaced by smaller constants.

The following theorem was treated in [14] and it obtains bounds in terms of the $L_p([a,b] \times [c,d])$ norms.

Theorem 10. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a continuous mapping on* $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$ *and is in* $L_p[(a,b) \times (c,d)]$, *i.e.,*

$$\left\| f''_{s,t} \right\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right|^p dx dy \right)^{\frac{1}{p}} < \infty, \qquad p \ge 1$$

then we have the inequality:

$$\left|\int_a^b \int_c^d f(s,t)\,dsdt - \left[(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(s,y)\,ds\right.\right. \\ \left.\left. - (d-c)(b-a) f(x,y)\right| \right. \tag{1.16}$$

$$\leq \begin{cases} \left[\dfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left[\dfrac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\|f''_{s,t}\right\|_p, \\ \qquad\qquad\qquad\qquad f''_{s,t} \in L_p\left[(a,b)\times(c,d)\right], \\ \left[\dfrac{1}{2}+\dfrac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\left[\dfrac{1}{2}+\dfrac{\left|y-\frac{c+d}{2}\right|}{d-c}\right](b-a)(d-c)\left\|f''_{s,t}\right\|_1, \\ \qquad\qquad\qquad\qquad f''_{s,t} \in L_1\left[(a,b)\times(c,d)\right], \end{cases}$$

for all $(x,y) \in [a,b]\times[c,d]$.

In the current article it is proposed to obtain Ostrowski type results for multidimensional integrals using an iterative approach using the one-dimensional result as a *seed* or *generator* of the multidimensional.

An identity for multidimensional integrals using an iterative approach is presented in Section 2 and bounds are obtained in terms of the $L_p[I^n]$ norms where $I^n = [a_1,b_1]\times\cdots\times[a_n,b_n]$. Perturbed rules are developed in Section 3 while Section 4 describes work underway to extend the results to involve higher derivatives, Trapezoidal type rules and three point rules.

The present work recaptures the results of Theorems 9 and 10 as particular instances. It allows the approximation of a multidimensional integral in terms of lower dimensional integrals and function evaluations.

2. Identities from an Iterative Approach

The following theorem uses an iterative approach to extend the Ostrowski functional identity to multidimensions. Firstly, we will require some notation.

Let $I^n = \prod_{i=1}^{n}[a_i,b_i] = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$. Further, let $f : I^n \to \mathbb{R}$ and define operators $F_i(f)$ and $\lambda_i(f)$ by

$$F_i(f) := f(t_1,\ldots,t_{i-1},x_i,t_{i+1},\ldots,t_n) \text{ where } x_i \in [a_i,b_i] \tag{2.1}$$

and

$$\lambda_i(f) := \frac{1}{d_i}\int_{a_i}^{b_i} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n)\,dt_i. \tag{2.2}$$

That is, $F_i(f)$ evaluates $f(\cdot)$ in the ith variable at $x_i \in [a_i,b_i]$ and $\lambda_i(f)$ is the integral mean of $f(\cdot)$ in the ith variable. Assuming that $f(\cdot)$ is absolutely continuous in the ith variable $t_i \in [a_i,b_i]$, we have

$$\mathcal{L}_i(f) := \frac{1}{d_i}\int_{a_i}^{b_i} p_i(x_i,t_i)\frac{\partial f}{\partial t_i}dt_i = (F_i-\lambda_i)(f), \tag{2.3}$$

for $i = 1, 2, \ldots, n$, where

$$\frac{p_i(x_i, t_i)}{d_i} = \begin{cases} \dfrac{t_i - a_i}{b_i - a_i}, & t_i \in [a_i, x_i] \\ \\ \dfrac{t_i - b_i}{b_i - a_i}, & t_i \in (x_i, b_i], \end{cases} \tag{2.4}$$

and $d_i = b_i - a_i$.

Thus (2.3) – (2.4) is ostensibly equivalent to identity (1.4) for $f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)$ absolutely continuous for $t_i \in [a_i, b_i]$.

Theorem 11. *Let $f : I^n \to \mathbb{R}$ be absolutely continuous in such a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$\begin{aligned} &E_n(f) \\ = \;& f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} \frac{1}{d_i} \int_{a_i}^{b_i} f(x_1, x_2, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n)\, dt_i \\ & + \sum_{i<j}^{n} \frac{1}{d_i d_j} \int_{a_j}^{b_j} \int_{a_i}^{b_i} f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_j, \ldots, x_n)\, dt_i dt_j \\ & - \cdots\cdots\cdots - \frac{(-1)^n}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_i}^{b_i} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \end{aligned} \tag{2.5}$$

where

$$E_n(f) := \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{n} p_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \ldots \partial t_1} dt_1 \ldots dt_n, \tag{2.6}$$

$$D_n = \prod_{i=1}^{n} d_i, \quad d_i = b_i - a_i, \tag{2.7}$$

and $p_i(x_i, t_i)$ is given by (2.4).

Proof. Define $E_r(f)$ by

$$E_r(f) = \left(\prod_{i=1}^{r} \mathcal{L}_i \right)(f) \tag{2.8}$$

then from the left identity in (2.3), $E_n(f)$ is as given by (2.6). Further,

$$E_r(f) = \mathcal{L}_r(E_{r-1}(f)), \quad \text{for} \quad r = 1, 2, \ldots, n \tag{2.9}$$

where $E_0(f) = f$.

Now, from (2.8),

$$E_1(f) = \mathcal{L}_1(f) = (F_1 - \lambda_1)(f),$$

which is the Montgomery identity for $t_1, x_1 \in [a_1, b_1]$

$$\begin{aligned} E_1(f) &= \frac{1}{d_1} \int_{a_1}^{b_1} p_1(x_1, t_1) \frac{\partial f}{\partial t_1}(t_1, t_2, \ldots, t_n)\, dt_1 \\ &= f(x_1, t_2, \ldots, t_n) - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, t_2, \ldots, t_n)\, dt_1. \end{aligned} \tag{2.10}$$

Further,

$$\begin{aligned}
E_2(f) &= \mathcal{L}_2(E_1(f)) = (F_2 - \lambda_2)(E_1(f)) \\
&= F_2(E_1(f)) - \lambda_2(E_1(f)) \\
&= f(x_1, x_2, t_3, \ldots, t_n) - \frac{1}{d_1}\int_{a_1}^{b_1} f(t_1, x_2, t_3, \ldots, t_n)\,dt_1 \\
&\quad -\frac{1}{d_2}\int_{a_2}^{b_2}\left[f(x_1, t_2, \ldots, t_n) - \frac{1}{d_1}\int_{a_1}^{b_1} f(t_1, t_2, \ldots, t_n)\,dt_1\right]dt_2 \\
&= f(x_1, x_2, t_3, \ldots, t_n) - \frac{1}{d_1}\int_{a_1}^{b_1} f(t_1, x_2, t_3, \ldots, t_n)\,dt_1 \\
&\quad -\frac{1}{d_2}\int_{a_2}^{b_2} f(t_1, t_2, t_3, \ldots, t_n)\,dt_2 + \frac{1}{d_1 d_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1, t_2, \ldots, t_n)\,dt_1 dt_2
\end{aligned}$$

and continuing in this manner until $r = n$ gives the result as stated in (2.5). ∎

Remark 1. *The result given by (2.5) may be utilised to approximate the $n-$ dimensional integral in terms of lower dimensional integrals and a function evaluation $f(x_1, x_2, \ldots, x_n)$ where $x_i \in [b_i, a_i]$, $i = 1, 2, \ldots, n$. Specifically, there are $\binom{n}{0}$ function evaluations, $\binom{n}{1}$ single integral evaluations etc., in each of the axes, $\binom{n}{2}$ double integral evaluations and so on, and, of course, $\binom{n}{n}$ $n-$dimensional integral evaluations. This results from the fact that from (2.8) and (2.1) – (2.3)*

$$E_n(f) = \left(\prod_{i=1}^{n} \mathcal{L}_i\right)(f) = \left(\prod_{i=1}^{n} (F_i - \lambda_i)\right)(f). \tag{2.11}$$

It will be subsequently demonstrated that the above procedure of utilising a one-dimensional identity as the *seed* or *generator* to recursively obtain a multidimensional identity may be extended to other seed identities.

The following theorem bounds for $\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right)$ are obtained where

$$\begin{aligned}
(2.12)\quad & \tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right) \\
= \quad & f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} \frac{1}{d_i}\int_{a_i}^{b_i} f(x_1, x_2, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n)\,dt_i \\
& + \sum_{i<j}^{n} \frac{1}{d_j d_i}\int_{a_j}^{b_j}\int_{a_i}^{b_i} f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_n)\,dt_i dt_j \\
& - \cdots\cdots\cdots - \frac{(-1)^n}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(t_1, \ldots, t_n)\,dt_1 \ldots dt_n
\end{aligned}$$

and $\underset{\sim}{z} = (z_1, z_2, \ldots, z_n)$.

Theorem 12. *Let the conditions of Theorem 11 continue to hold. Then*

$$\left|\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right)\right| \tag{2.13}$$

$$\leq \begin{cases} \prod_{i=1}^{n} P_i\,(1) \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{\infty}, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_\infty\,[I^n]\,; \\ \left(\prod_{i=1}^{n} P_i\,(q) \right)^{\frac{1}{q}} \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{p}, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_p\,[I^n]\,, \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \prod_{i=1}^{n} \theta_i \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{1}, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_1\,[I^n]\,, \end{cases} \tag{2.14}$$

where $\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right)$ *is as defined by (2.12),*

$$(q+1)\,P_i\,(q) = (x_i - a_i)^{q+1} + (b_i - x_i)^{q+1}\,, \tag{2.15}$$

$$\theta_i = \frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right|. \tag{2.16}$$

Proof. From (2.6) and (2.12)

$$\begin{aligned} & \left| \tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right) \right| \\ = \quad & |E_n\,(f)| \leq \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^{n} p_i\,(x_i, t_i)\, \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right| dt_1 \ldots dt_n. \end{aligned} \tag{2.17}$$

Now, for $\frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_\infty\,[I^n]$, then

$$\begin{aligned} & D_n\,|E_n\,(f)| \\ \leq \quad & \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^{n} p_i\,(x_i, t_i) \right| dt_1 \ldots dt_n \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{\infty} \\ = \quad & \prod_{i=1}^{n} \int_{a_i}^{b_i} |p_i\,(x_i, t_i)|\, dt_i \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{\infty} \\ = \quad & \prod_{i=1}^{n} \left[\int_{a_i}^{x_i} (t_i - a_i)\, dt_i + \int_{x_i}^{b_i} (b_i - t_i)\, dt_i \right] \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{\infty} \\ = \quad & \frac{1}{2^n} \prod_{i=1}^{n} \left[(x_i - a_i)^2 + (b_i - x_i)^2 \right] \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{\infty}. \end{aligned} \tag{2.18}$$

Hence combining (2.17) and (2.18) gives the first inequality of (2.13).

Further, using the Hölder inequality we have for $\frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_p\,[I^n]$, $1 \leq p < \infty$,

$$D_n\,|E_n\,(f)| \leq \left(\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^{n} p_i\,(x_i, t_i) \right|^q dt_1 \ldots dt_n \right)^{\frac{1}{q}} \left\| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right\|_{p},$$

where

$$\begin{aligned}
& \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^{n} p_i(x_i, t_i) \right|^q dt_1 \ldots dt_n \\
= & \prod_{i=1}^{n} \int_{a_i}^{b_i} |p_i(x_i, t_i)|^q \, dt_i \\
= & \prod_{i=1}^{n} \left[\int_{a_i}^{x_i} (t_i - a_i)^q \, dt_i + \int_{x_i}^{b_i} (b_i - t_i)^q \, dt_i \right] \\
= & \frac{1}{(q+1)^n} \prod_{i=1}^{n} \left[(x_i - a_i)^{q+1} + (b_i - x_i)^{q+1} \right]
\end{aligned}$$

and so the second inequality is valid on noting (2.15).

The final inequality in (2.13) is obtained from (2.17) for $\frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_1[I^n]$, giving

$$\begin{aligned}
D_n |E_n(f)| & \leq \sup_{\underset{\sim}{t} \in \left[\underset{\sim}{a}, \underset{\sim}{b} \right]} \left| \prod_{i=1}^{n} p_i(x_i, t_i) \right| \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \right| dt_1 \ldots dt_n \\
& = \prod_{i=1}^{n} \sup_{t_i \in [a_i, b_i]} |p_i(x_i, t_i)| \left\| \frac{\partial f^n}{\partial t_n \ldots \partial t_1} \right\|_1 \\
& = \prod_{i=1}^{n} \max\{x_i - a_i, b_i - x_i\} \left\| \frac{\partial f^n}{\partial t_n \ldots \partial t_1} \right\|_1 .
\end{aligned}$$

On noting that $\max\{X, Y\} = \frac{X+Y}{2} + \frac{|X-Y|}{2}$ readily produces the stated result. ∎

Remark 2. *The expression for* $\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right)$ *may be written in a less explicit form which is perhaps more appealing. Namely,*

$$\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right) = f(x_1, x_2, \ldots, x_n) + \sum_{k=1}^{n-1} (-1)^k \sum_{k} \mathcal{M}_k + (-1)^n \mathcal{M}_n, \tag{2.19}$$

where $\mathcal{M}_k$ *represents the integral means in* k *variables with the remainder being evaluated at their respective interior point and* $\sum_k \mathcal{M}_k$ *is a sum over all* $\binom{n}{k}$, $k-$*dimensional integral means. Here*

$$\mathcal{M}_n = \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(t_1, \ldots, t_n) \, dt_1 \ldots dt_n$$

and

$$\begin{aligned}
\sum_{1} \mathcal{M}_1 & = \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, x_2, \ldots, x_n) \, dt_1 + \frac{1}{d_2} \int_{a_2}^{b_2} f(x_1, t_2, x_3, \ldots, x_n) \, dt_2 \\
& + \cdots + \frac{1}{d_n} \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_{n-1}, t_n) \, dt_n.
\end{aligned}$$

Corollary 1. *Let the conditions of Theorem 11 hold and let* $\alpha_i = \frac{a_i+b_i}{2}$*, then*

$$\left|\tau_n\left(\underset{\sim}{a},\underset{\sim}{\alpha},\underset{\sim}{b}\right)\right| \tag{2.20}$$
$$\leq \begin{cases} \frac{D_n}{2^n}\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_\infty, & \frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_\infty\left[I^n\right]; \\ \frac{D_n^{1+\frac{1}{q}}}{2^n\left(q+1\right)^{\frac{n}{q}}}\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_p, & \frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_p\left[I^n\right], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{D_n}{2^n}\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_1, & \frac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_1\left[I^n\right], \end{cases}$$

where $\underset{\sim}{\alpha} = \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$, $\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$ *is as given by (2.12) or (2.19) and*

$$D_n = \prod_{i=1}^{n} d_i = \prod_{i=1}^{n}\left(b_i - a_i\right).$$

Proof. Taking $x_i = \frac{a_i+b_i}{2}$, that is $\underset{\sim}{x} = \underset{\sim}{\alpha}$ in Theorem 12 produces the tightest bound from (2.13) as given by (2.20). ∎

Remark 3. *We note that taking* $n=1$ *in (2.13) produces the results (1.5) which were obtained by Dragomir and Wang* [19] – [21]*. If* $n=2$ *is taken, then the first inequality in (2.13) reproduces the results of Barnett and Dragomir* [3] *as represented in Theorem 9, equation (1.15). The results (1.16) are obtained from the remainder of the inequalities in (2.13). Double integral Ostrowski type results have also been examined in* [16]*.*

3. Perturbed Rules from the Chebychev Functional

For $g,h : [a,b] \to \mathbb{R}$ the following $\mathfrak{T}(g,h)$ is well known as the Chebychev functional. Namely,

$$\mathfrak{T}(g,h) = \mathcal{M}(gh) - \mathcal{M}(g)\,\mathcal{M}(h), \tag{3.1}$$

where $\mathcal{M}(g) = \frac{1}{b-a}\int_a^b g(t)\,dt$ is the integral mean (see [4], [5] and [7]).

The Chebychev functional (3.1) is known to satisfy a number of identities including

$$\mathfrak{T}(g,h) = \frac{1}{b-a}\int_a^b h(t)\left[g(t) - \mathcal{M}(g)\right]dt. \tag{3.2}$$

Further, a number of sharp bounds for $|\mathfrak{T}(g,h)|$ exist, under various assumptions about g and h, including (see [7] for example, or [25, p. 296])

$$|\mathfrak{T}(g,h)| \leq \begin{cases} \left[\mathfrak{T}(g,g)\right]^{\frac{1}{2}}\left[\mathfrak{T}(h,h)\right]^{\frac{1}{2}}, & g,h\in L_2[a,b] \\ \frac{A_u - A_l}{2}\left[\mathfrak{T}(h,h)\right]^{\frac{1}{2}}, & A_l \leq g(t) \leq A_u,\ t\in[a,b] \\ \left(\frac{A_u - A_l}{2}\right)\left(\frac{B_u - B_l}{2}\right), & B_l \leq h(t) \leq B_u,\ t\in[a,b]\ \text{(Grüss)}. \end{cases} \tag{3.3}$$

It will be demonstrated how (3.1) may be used to obtain perturbed results for which (3.2) will provide an identity with which to obtain bounds.

The multidimensional versions of the above results hold since they have been shown to hold for a general linear functional by Andrica and Badea [2] (see also [25, Chapters IX and X, p. 239 – 210]).

The following theorem gives perturbed results using a multidimensional version of (3.3).

Theorem 13. *Let the conditions of Theorem 12 hold. Then*

$$\left|\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)-\prod_{i=1}^{n}\left(x_i-\frac{a_i+b_i}{2}\right)\cdot\frac{1}{D_n}\left(\prod_{i=1}^{n}\Delta_i\right)(f)\right| \tag{3.4}$$

$$\leq\begin{cases}\kappa_n\left[\dfrac{1}{D_n}\left\|\dfrac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_2^2-\dfrac{1}{D_n^2}\left[\left(\prod\limits_{i=1}^{n}\Delta_i\right)(f)\right]^2\right]^{\frac{1}{2}}, & \dfrac{\partial^n f}{\partial t_n\ldots\partial t_1}\in L_2\left[I^n\right];\\[2ex] \kappa_n\left(\dfrac{B_U\left(\underset{\sim}{a},\underset{\sim}{b}\right)-B_L\left(\underset{\sim}{a},\underset{\sim}{b}\right)}{2}\right), & \\ \qquad B_L\left(\underset{\sim}{a},\underset{\sim}{b}\right)\leq\dfrac{\partial^n f}{\partial t_n\ldots\partial t_1}\leq B_U\left(\underset{\sim}{a},\underset{\sim}{b}\right),\ \underset{\sim}{t}\in I^n & \\[2ex] \left(\dfrac{A_U\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)-A_L\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)}{2}\right)\left(\dfrac{B_U\left(\underset{\sim}{a},\underset{\sim}{b}\right)-B_L\left(\underset{\sim}{a},\underset{\sim}{b}\right)}{2}\right), & \\ \qquad A_L\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)\leq P_n\left(\underset{\sim}{x},\underset{\sim}{t}\right)\leq A_U\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right),\quad \underset{\sim}{t}\in I^n, & \end{cases} \tag{3.5}$$

where,

$\tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)$ *is as defined by (2.12) or (2.19),*

D_n *is as given by (2.7),*

$$P_n\left(\underset{\sim}{x},\underset{\sim}{t}\right)=\prod_{i=1}^{n}p_i\left(x_i,t_i\right),\ p_i\left(x_i,t_i\right)\quad \textit{is as defined by (2.4)} \tag{3.6}$$

and

$$\Delta_i\left(f\right)=f\left(t_1,\ldots,t_{i-1},b_i,t_{i+1},\ldots,t_n\right)-f\left(t_1,\ldots,t_{i-1},a_i,t_{i+1},\ldots,t_n\right). \tag{3.7}$$

Proof. From (2.6) and (2.12) or (2.19)

$$\begin{aligned} E_n\left(f\right) &= \frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\prod_{i=1}^{n}p_i\left(x_i,t_i\right)\frac{\partial^n f}{\partial t_n\ldots\partial t_1}dt_1\ldots dt_n \\ &= \tau_n\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right) \end{aligned} \tag{3.8}$$

and we require to evaluate

$$G_n \quad : \quad = \frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\prod_{i=1}^{n} p_i\left(x_i, t_i\right) dt_1 \ldots dt_n \tag{3.9}$$
$$= \frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} P_n\left(\underset{\sim}{x}, \underset{\sim}{t}\right) dt_1 \ldots dt_n$$

where (3.6) has been used and,

$$H_n := \frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{\partial^n f}{\partial t_n \ldots \partial t_1} dt_1 \ldots dt_n \tag{3.10}$$

in order to calculate

$$\left|\tau_n\left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}\right) - G_n H_n\right|. \tag{3.11}$$

Here we associate $\mathcal{M}(g)$ and H_n with $\mathcal{M}(h)$ in (3.1) – (3.3).

Now, utilising (2.4),

$$G_n = \frac{1}{D_n}\prod_{i=1}^{n}\int_{a_i}^{b_i} p_i\left(x_i, t_i\right) dt_i = \frac{1}{D_n}\prod_{i=1}^{n}\left[\int_{a_i}^{x_i}\left(t_i - a_i\right) dt_i + \int_{x_i}^{b_i}\left(b_i - t_i\right) dt_i\right]$$
$$= \frac{1}{D_n}\prod_{i=1}^{n}\frac{A_i^2 - B_i^2}{2},$$

where $A_i = x_i - a_i$ and $B_i = b_i - x_i$.

Hence

$$G_n = \frac{1}{D_n}\prod_{i=1}^{n}\frac{\left(A_i + B_i\right)\left(A_i - B_i\right)}{2} = \frac{1}{D_n}\prod_{i=1}^{n} d_i\left(x_i - \frac{a_i + b_i}{2}\right) \tag{3.12}$$
$$= \prod_{i=1}^{n}\left(x_i - \frac{a_i + b_i}{2}\right).$$

Further, explicit evaluation of the integrals in (3.10) gives

$$H_n = \frac{1}{D_n}\left(\prod_{i=1}^{n}\Delta_i\right)(f), \tag{3.13}$$

where $\Delta_i(t)$ is as defined by (3.7). Substitution of (3.12) and (3.13) into (3.11) gives the left hand side of (3.4).

Now for the right hand side.

We require to calculate

$$\kappa_n^2 = G_n^{(2)} - G_n^2, \tag{3.14}$$

where

$$
\begin{aligned}
G_n^{(2)} &= \frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left(\prod_{i=1}^{n} p_i(x_i,t_i)\right)^2 dt_1\ldots dt_n \\
&= \frac{1}{D_n}\prod_{i=1}^{n}\int_{a_i}^{b_i} p_i^2(x_i,t_i)\,dt_i \\
&= \frac{1}{D_n}\prod_{i=1}^{n}\left[\int_{a_i}^{x_i}(t_i-a_i)^2\,dt_i+\int_{x_i}^{b_i}(t_i-b_i)^2\,dt_i\right] \\
&= \frac{1}{3D_n}\prod_{i=1}^{n}\left[A_i^3+B_i^3\right].
\end{aligned}
$$

Now $X^3+Y^3=(X+Y)\left[\left(\frac{X+Y}{2}\right)^2+3\left(\frac{X-Y}{2}\right)^2\right]$ and so

$$
\begin{aligned}
G_n^{(2)} &= \frac{1}{3D_n}\prod_{i=1}^{n} d_i\left[\left(\frac{d_i}{2}\right)^2+3\left(x_i-\frac{a_i+b_i}{2}\right)^2\right] \\
&= \frac{1}{3}\prod_{i=1}^{n}\left[\left(\frac{d_i}{2}\right)^2+3\left(x_i-\frac{a_i+b_i}{2}\right)^2\right].
\end{aligned}
\tag{3.15}
$$

Substituting (3.15) and (3.12) into (3.14) gives the first inequality in (3.4) on noting that

$$
\left[\frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left(\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right)^2 dt_1\ldots dt_n-H_n^2\right]^{\frac{1}{2}}=\left[\frac{1}{D_n}\left\|\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right\|_2^2-H_n^2\right]^{\frac{1}{2}}
$$

and using (3.13).

Now for the second inequality we note that (see [10], for example)

$$
\begin{aligned}
0 &\le \left\{\frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\left(\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\right)^2 dt_1\ldots dt_n \right.\\
&\qquad\left. -\left[\frac{1}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{\partial^n f}{\partial t_n\ldots\partial t_1}dt_1\ldots dt_n\right]^2\right\}^{\frac{1}{2}} \\
&\le \frac{B_U\left(\underset{\sim}{a},\underset{\sim}{b}\right)-B_L\left(\underset{\sim}{a},\underset{\sim}{b}\right)}{2},\quad\text{where}\quad B_L\left(\underset{\sim}{a},\underset{\sim}{b}\right)\le\frac{\partial^n f}{\partial t_n\ldots\partial t_1}\le B_U\left(\underset{\sim}{a},\underset{\sim}{b}\right).
\end{aligned}
$$

The second inequality is thus coarser than the first, but may be more useful in certain implementations.

The final inequality is obtained on noting that

$$
\begin{aligned}
0\le\kappa_n &\le \frac{A_U\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)-A_L\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right)}{2}, \\
A_L\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right) &\le P_n\left(\underset{\sim}{x},\underset{\sim}{t}\right)\le A_U\left(\underset{\sim}{a},\underset{\sim}{x},\underset{\sim}{b}\right),\quad \underset{\sim}{t}\in I^n,
\end{aligned}
$$

where κ_n is as defined by (3.14). The inequalities in (3.4) are in increasing coarseness. ∎

Remark 4. *It is well known that (see* [25, p. 296]*) for* $\phi \leq h(t) \leq \Phi$*:*

$$\begin{aligned} 0 &\leq [\mathfrak{T}(h,h)]^{\frac{1}{2}} \leq \left[\frac{1}{b-a}\|h\|_2^2 - \mathcal{M}^2(h)\right]^{\frac{1}{2}} \\ &\leq (\Phi - \mathcal{M}(h))(\mathcal{M}(h) - \phi) \leq \left(\frac{\Phi-\phi}{2}\right)^2. \end{aligned}$$

The second inequality may not be as useful since if h *were the kernel, then we would be able to evaluate* $[\mathcal{T}(h,h)]^{\frac{1}{2}}$ *explicitly. If it were the derivative of the integrand, then* $\mathcal{M}(h)$ *would be the difference operator for example, (3.13). Thus a bound between the first and second in (3.4) could be*

$$\kappa_n\left(B_U\left(\underset{\sim}{a},\underset{\sim}{b}\right) - H_n\right)\left(H_n - B_L\left(\underset{\sim}{a},\underset{\sim}{b}\right)\right),$$

where H_n *is as given by (3.10) or explicitly as (3.13).*

4. Concluding Remarks and Discussion

The current paper has demonstrated how to obtain identities and subsequent bounds for multidimensional integrals using one dimensional Ostrowski results as the *generator* or *seed*. This is accomplished in an iterative manner.

Many other seed or generator results may be utilised to extend the problems to multidimensional formulations.

The generalised trapezoidal rule

$$T(f;c,x,d) := \left(\frac{x-c}{d-c}\right)f(c) + \left(\frac{d-x}{d-c}\right)f(d) - \mathcal{M}(f;c,d) \tag{4.1}$$

satisfies the identity

$$T(f;c,x,d) = \int_c^d q(x,t,c,d)\,df(t), \quad q(x,t,c,d) = \frac{t-x}{d-c}, \quad x,t \in [c,d], \tag{4.2}$$

as shown in [5].

Further, define the three point functional $\mathfrak{T}(f;a,\alpha,x,\beta,b)$ which involves the difference between the integral mean and, a weighted combination of a function evaluated at the end points and an interior point. Namely, for $a \leq \alpha < x < \beta \leq b$,

$$\begin{aligned} \mathfrak{T}(f;a,\alpha,x,\beta,b) \quad : \quad &= \left(\frac{\alpha-a}{b-a}\right)f(a) + \left(\frac{\beta-\alpha}{b-a}\right)f(x) \\ &+ \left(\frac{b-\beta}{b-a}\right)f(b) - \mathcal{M}(f;a,b). \end{aligned} \tag{4.3}$$

Cerone and Dragomir [6] showed that for f of bounded variation, the identity

$$\mathfrak{T}(f;a,\alpha,x,\beta,b) = \int_a^b r(x,t)\,df(t), \quad r(x,t) = \begin{cases} \dfrac{t-\alpha}{b-a}, & t \in [a,x] \\[2ex] \dfrac{t-\beta}{b-a}, & t \in (x,b] \end{cases} \tag{4.4}$$

is valid. They effectively demonstrated that the Ostrowski functional and the trapezoid functional could be recaptured as particular instances. Specifically, from (4.3) and (4.4),

$$S(f;a,x,b) = \mathfrak{T}(f;a,a,x,b,b)$$

and

$$T(f;a,x,b)=\mathfrak{T}(f;a,x,x,x,b),$$

where $S(f;a,x,b)$ and $T(f;a,x,b)$ are defined by (1.1) and (4.1) and satisfy identities (1.3) and (4.2) respectively.

It should be noted at this stage that

$$\begin{aligned}&(b-a)\,\mathfrak{T}\left(f;a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\\ =\ &\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(x)\,dx\end{aligned}$$

is the Simpson functional.

Further, if $f(t)$ is assumed to be absolutely continuous for t over its respective interval, then $df(t)=f'(t)\,dt$ and the Riemann-Stieltjes integrals in (4.2) and (4.4) are equivalent to Riemann integrals.

Work is proceeding to utilise the generalised trapezoidal rule and three point rules defined by (4.1) – (4.4) as generators for multidimensional integrals.

The work may also be extended to allow the error to be expressed in terms of the higher derivatives. Thus, for example, the identity of Cerone et al. [9]

$$\begin{aligned}&(-1)^m\int_a^b P_m(x,t)\,f^{(m)}(t)\,dt\\ =\ &\int_a^b f(t)\,dt-\sum_{k=1}^{n}\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right]\frac{f^{(k-1)}(x)}{k!},\end{aligned}\tag{4.5}$$

where

$$P_m(x,t)=\begin{cases}\dfrac{(t-a)^m}{m!}, & t\in[a,x]\\[2ex] \dfrac{(t-b)^m}{m!}, & t\in(x,b]\end{cases}\tag{4.6}$$

may be used as a generator for $n-$dimensional integrals. Hanna et al. [23] utilise (4.5) – (4.6) to obtain approximations to double integrals in terms of single integrals and, function and lower order derivative evaluations at interior points. Double integrals have also been examined in [16]. An operatorial manner is being investigated to obtain an $m\times n$ recursive formulation of the problem for $n-$dimensional integrals involving the $m-$derivative in each direction.

Finally, we are not restricted to using the same identity in each of the directions but may use different ones as long as one is able to justify this.

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