(g, h, M) convex sets. The problem of the best approximation

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Abstract. In this paper the problem of the best approximation of an element of the space $\mathbf{R^n}$ by elements of a (g, h, M) convex set is discussed.

Key Words: element of the best approximation, generalized convexity.

1 Introduction

Let be $X \subseteq \mathbf{R}^n$, $X \neq \emptyset$, and $y^0 \in \mathbf{R}^n$.

Definition 1.1. *A point $x^0 \in X$ is called element of the best approximation of y^0 by elements of X if*

$$\| y^0 - x^0 \| \leq \| y^0 - x \|, \quad \textit{for all} \quad x \in X. \tag{1}$$

It is known that (see for example [2] or [5]):

Theorem 1.1. *If $X \subseteq \mathbf{R}^n$ is a convex set and if y^0 is a given point of $\mathbf{R}^n$, then there exists at most one element of the best approximation of y^0 by elements of X.*

In the following, we show that this property of convex sets remains true if the set X is not convex, but is (g, h, M) convex and if some additional hypothesis are fulfilled.

Let n be a natural number. If $h = (h_1, ..., h_n) \in B^n$, where $B = \{0, 1\}$, then we put

$$|h| = h_1 + ... + h_n.$$

We consider the function $\Lambda = (\Lambda_1, ..., \Lambda_n) : \mathbf{R}^n \to \mathbf{R}^n$, given, for each $i \in \{1, ..., n\}$ by

$$\Lambda_i(x) = \begin{cases} x_{n-|h|+\sum_{k=1}^{i} h_k}, & \text{if} \quad h_i = 1, \\ x_1, & \text{if} \quad h_i = 0 \quad \text{and} \quad i = 1, \\ x_{i-\sum_{k=1}^{i-1} h_k}, & \text{if} \quad h_i = 0 \quad \text{and} \quad i > 1, \end{cases}$$

for all $x = (x_1, ..., x_n) \in \mathbf{R}^n$.

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For $A \subseteq \mathbf{R}^n$ and $h = (h_1, ..., h_n) \in B^n$, we denote by

$$pr(h, A) = \begin{cases} A, \quad \text{if} \quad |h| = 0 \\ \{y \in \mathbf{R}^{n-|h|} \mid \exists z \in \mathbf{R}^{|h|} \text{ with } \Lambda(y, z) \in A\}, \text{ if } 0 < |h| < n \\ \emptyset, \quad \text{if} \quad |h| = n. \end{cases}$$

If $|h| \neq n$, for each $y \in pr(h, A)$, we put

$$s(y, h, A) = \begin{cases} \emptyset, \quad |h| = 0 \\ \{z \in \mathbf{R}^{|h|} \mid \Lambda(y, z) \in A\}, \quad \text{if} \quad 0 < |h| < n. \end{cases}$$

Definition 1.1. *A set $A \subseteq \mathbf{R}^n$ is said to be (g, h, M) convex if*
i) *$pr(h, A) \neq \emptyset$ and*

$$M \bigcap \{\Lambda(y, z) \mid z \in conv(g(s(y, h, A)))\} \subseteq \{\Lambda(y, z) \mid z \in s(y, h, A)\},$$

for all $y \in pr(h, A)$; or
ii) $pr(h, A) = \emptyset$ and $M \bigcap conv(g(A)) \subseteq A$.

In this paper, by $1_{\mathbf{R}^n}$, we denote the identity function

$$1_{\mathbf{R}^n} : \mathbf{R}^n \to \mathbf{R}^n, \quad 1_{\mathbf{R}^n}(x) = x, \quad \text{for all} \quad x \in \mathbf{R}^n,$$

and we put $e_{\mathbf{R}^n} = (1, ..., 1) \in B^n$,

Remark 1.1. Let n be a natural number, $A \subseteq \mathbf{R}^n$, M a nonvoid subset of $\mathbf{R}^n$, and let $E : \mathbf{R}^n \to \mathbf{R}^n$ be a function. The following propositions are true:

- The set A is convex in the classical sens if and only if it is $(1_{\mathbf{R}^n}, e_{\mathbf{R}^n}, \mathbf{R}^n)$ convex.
- The set A is E-convex (see [7]) if and only if it is $(E, e_{\mathbf{R}^n}, \mathbf{R}^n)$ convex.
- The set A is strongly convex with respect to M (see [4]) if and only if it is $(1_{\mathbf{R}^n}, e_{\mathbf{R}^n}, M)$ convex.

From Remark 1.1 it results that the set of (g, h, M) convex sets strictly includes the set of convex sets (in the classical sens), the set of convex sets with respect to a given set and the set of E-convex sets.

2 The case of $(g, e_{\mathbf{R}^n}, \mathbf{R}^n)$ convex sets

We remark that a subset $A \subseteq \mathbf{R}^\mathbf{n}$ is $(g, e_{\mathbf{R}^n}, \mathbf{R}^n)$ convex if and only if

- $A = \emptyset$, or
- $A \neq \emptyset$, and we have

$$conv(g(A)) \subseteq A \tag{2}$$

i.e. A is E-convex in the sens of definition in [7] (if we take $E = g$).

Theorem 2.1 *If $A \subseteq \mathbf{R}^n$ is a nonempty $(g, e_{\mathbf{R}^n}, \mathbf{R}^n)$ convex set and if $x^0 \in g(A)$ is an element of the best approximation of $y^0 \in \mathbf{R}^n$ by elements of A, then x^0 is the single element of the best approximation of y^0 by elements of $g(A)$.*

Proof. From (2) we have

$$g(A) \subseteq conv(g(A)) \subseteq A. \tag{3}$$

As $x^0 \in g(A)$ is an element of the best approximation of y^0 by elements of A, we get that x^0 is an element of the best approximation of y^0 by elements of $conv(g(A))$. In view of theorem 1.1 it is the single element of the best approximation of y^0 by elements of $conv(g(A))$. Relation (3) implies that it is also the single element of the best approximation of y^0 by elements of $g(A)$.

3 The case of $(g, e_{\mathbf{R}^n}, M)$ convex sets

Theorema 3.1 *If $A \subseteq \mathbf{R}^n$ is a $(g, 1_{\mathbf{R}^n}, M)$ convex set and if $x^0 \in (g(A) \bigcap int\, M)$ is an element of the best approximation of $y^0 \in (M \setminus A)$ by elements of A, then x^0 is the single element of the best approximation of y^0 by elements of $g(A)$.*

Proof. Let us suppose that there exists an $y \in g(A)$, $y \neq x^0$, such that

$$\| y^0 - x^0 \| = \| y^0 - y \| \neq 0. \tag{4}$$

As $x^0 \in (g(A) \bigcap int\, M)$, there exists a real number $t \in]0, 1[$ such that

$$z = tx^0 + (1-t)y \in M. \tag{5}$$

On the other hand, easily we get

$$z \in A. \tag{6}$$

($x^0 \in g(A)$ and $y \in g(A)$ implies that there exist $x', x" \in A$ such that

$$z = tx^0 + (1-t)y = tg(x') + (1-t)g(x");$$

as A is $(g, e_{\mathbf{R}^n}, M)$ convex, it results that $z \in A$.)

We have

$$\| z - y^0 \| \leq t \| x^0 - y^0 \| + (1-t) \| y - y^0 \| = \| x^0 - y^0 \| . \tag{7}$$

Since x^0 is an element of the best approximation of y^0 by elements of A, the inequality (7) can not be a strict one. Then

$$\| z - y^0 \| = t \| x^- - y^0 \| + (1-t) \| y - y^0 \| = \| x^0 - y^0 \| . \tag{8}$$

It follows that

$$\| t(x^0 - y^0) \| + \| (1-t)(y - y^0) \| = \| x^0 - y^0 \| . \tag{9}$$

Applying the Cauchy-Schwarz inequality we get that there exist two real numbers a and b with

$$\mid a \mid + \mid b \mid \neq 0, \tag{10}$$

such that

$$at(x)k^0 - y_k^0) + b(1-t)(y_k - y_k^0) = 0, \quad k \in \{1, ..., n\}. \tag{11}$$

Since $x^0 \neq y^0$ and $y \neq y^0$, it follows that $a \neq 0$ and $b \neq 0$. Two cases may appear:

$$i)\ a \cdot b > 0 \quad \text{or} \quad ii)\ a \cdot b < 0.$$

$i)$ If $a \cdot b > 0$, then (11) implies

$$y^0 = \frac{at}{at + (1-t)b} x^0 + \frac{b(1-t)}{at + (1-t)b} y =$$

$$= \frac{at}{at + (1-t)b} g(x') + \frac{b(1-t)}{at + (1-t)b} g(x") \in conv(g(A)) \bigcap M \subseteq A.$$

This is a contradiction.

$ii)$ If $a \cdot b < 0$, then from (11) we get

$$\| x^0 - y^0 \| = \mid \frac{a}{b} \cdot \frac{1-t}{t} \cdot \| y - y^0 \| .$$

As $\| x^0 - y^0 \| = \| y - y^0 \|$, it results $t = \mid b \mid (\mid a \mid + \mid b \mid)^{-1}$. Replacing t in (11), we get

$$a \mid b \mid x^0 + b \mid a \mid y - (a \mid b \mid + b \mid a \mid) y^0 = 0. \tag{12}$$

As $a \cdot b < 0$, we obtain $a \mid b \mid + b \mid a \mid = 0$. Then, from (12), it follows $x^0 = y$, which is a contradiction.

Since in both cases a contradiction comes across, the assumption that x^0 is not a single element of the best approximation of y^0 by elements of $g(A)$ is false.$\diamond$

4 The general case of (g, h, M) convex sets

Theorem 4.1. *If $A \subseteq \mathbf{R}^n$ is a (g, h, M) convex set and $x^0 \in \mathbf{R}^n$, then the set $A(x^0)$ of all the elements of the best approximation of x^0 by elements of A is also a (g, h, M) convex set.*

Proof. If $card\, A(x^0) \in \{0, 1\}$; obviously $A(x^0)$ is a (g, h, M) convex set.

Let now $card\, A(x^0) > 1$. We remark that, in this case, we have $pr(h, A) \neq \emptyset$; otherwise, in the same way what we used to prove Theorem 3.1, it is possible to show that if $pr(h, A) = \emptyset$, then $card\, A(x^0) \in \{0, 1\}$.

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