ON AN EXTENSION OF HARDY-HILBERT'S INEQUALITY

BICHENG YANG

Abstract. In this paper, by introducing a parameter λ, we give a new extension of Hardy-Hilbert's inequality with a best possible constant factor. We also consider its equivalent form and the corresponding extended integral form.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{a} = 1$, a_n, $b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible (see [1]). (1.1) is well known as Hardy-Hilbert's inequality, which is important in analysis and its applications (see [2]). Its equivalent form is,

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m b_m}{m+n} \right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \sum_{n=1}^{\infty} a_n^p, \tag{1.2}$$

where the constant $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p$ is still the best possible. The corresponding integral form of (1.1) and (1.2) are:

If $f, g \geq 0$, $0 < \int_0^{\infty} f^p(t)\, dt < \infty$, and $0 < \int_0^{\infty} g^q(t)\, dt < \infty$, then

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)\, g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^{\infty} f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\infty} g^q(x)\, dx \right\}^{\frac{1}{q}}; \tag{1.3}$$

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^{\infty} f^p(x)\, dx, \tag{1.4}$$

where the constants $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p$ are all the best possible.

Date: Juune 23, 2001.
1991 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Hardy-Hilbert's inequality, weight coefficient, Holder's inequality.

In recent years, Gao and Yang [3] and Pachpatte [4] gave (1.1) some improvements. By introducing a parameter λ, Yang [5] gave a generalisation of (1.3) as

$$\begin{aligned}(1.5)\qquad & \int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}dxdy \\ & < k_\lambda(p)\left\{\int_0^\infty x^{1-\lambda} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{1-\lambda} g^q(x)\,dx\right\}^{\frac{1}{q}},\end{aligned}$$

where $\lambda > 2-\min\{p,q\}$, and the constant $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible ($B(u,v)$ is the $\beta-$function). Moreover, Kuang [6] gave another generalisation of (1.3) as

$$\begin{aligned}(1.6)\qquad & \int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x^\lambda+y^\lambda}dxdy \\ & < h_\lambda(p)\left\{\int_0^\infty x^{1-\lambda} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{1-\lambda} g^q(x)\,dx\right\}^{\frac{1}{q}},\end{aligned}$$

where $\max\left\{\frac{1}{p},\frac{1}{q}\right\} < \lambda \le 1$ and $h_\lambda(p) = \frac{\pi}{\left[\lambda \sin^{\frac{1}{p}}\left(\frac{\pi}{p\lambda}\right)\sin^{\frac{1}{q}}\left(\frac{\pi}{q\lambda}\right)\right]}$.

When $\lambda = 1$, both (1.5) and (1.6) change to (1.3).

In this paper, following the way of [5], we give a new extension of (1.1), with a best possible constant factor. We also consider its equivalent form and the corresponding extended integral form.

2. The Extended Series Form

Lemma 1. *For $s > 1$, $\frac{1}{s}+\frac{1}{r} = 1$, $\lambda > 0$, define the weight function $\omega_\lambda(y,s)$ as*

$$(2.1)\qquad \omega_\lambda(y,s) = \int_0^\infty \frac{1}{x^\lambda+y^\lambda}\left(\frac{y^{\frac{s}{r}}}{x}\right)^{1-\lambda}\left(\frac{y}{x}\right)^{\frac{\lambda}{r}}dx, \quad y\in(0,\infty),$$

and the weight coefficient $\tilde{\omega}_\lambda(n,s)$ as

$$(2.2)\qquad \tilde{\omega}_\lambda(n,s) = \sum_{m=1}^\infty \frac{1}{m^\lambda+n^\lambda}\left(\frac{n^{\frac{s}{r}}}{m}\right)^{1-\lambda}\left(\frac{n}{m}\right)^{\frac{\lambda}{r}}, \quad n\in\mathbb{N},$$

Then we have $\tilde{\omega}_\lambda(y,s) = \frac{y^{(1-\lambda)(s-1)}\pi}{\lambda\sin\left(\frac{\pi}{s}\right)}$, and for $0<\lambda\le s$, $n\in\mathbb{N}$, $\tilde{\omega}_\lambda(n,s) < \omega_\lambda(n,s)$.

Proof. Setting $t = \frac{x^\lambda}{y^\lambda}$ in (2.1), we find

$$\begin{aligned}\omega_\lambda(y,s) &= \frac{1}{\lambda}\int_0^\infty \frac{1}{y^\lambda(1+t)}\left(\frac{y^{\frac{s}{r}}}{yt^{\frac{1}{\lambda}}}\right)^{1-\lambda}\left(\frac{1}{t}\right)^{\frac{1}{r}} yt^{\frac{1}{\lambda}-1}dt \\ &= \frac{1}{\lambda}y^{(s-1)(1-\lambda)}\int_0^\infty \frac{1}{(1+t)}\left(\frac{1}{t}\right)^{\frac{1}{r}}dt \\ &= \frac{1}{\lambda}y^{(s-1)(1-\lambda)}\frac{\pi}{\sin\left(\frac{\pi}{s}\right)}.\end{aligned}$$

Since for $0 < \lambda \leq s$, $1 - \lambda + \frac{\lambda}{r} = 1 - \frac{\lambda}{s} \geq 0$, then the function

$$f(x) = \frac{1}{x^\lambda + n^\lambda} \left(\frac{n^{\frac{s}{r}}}{x} \right)^{1-\lambda} \left(\frac{n}{x} \right)^{\frac{\lambda}{r}} = \frac{1}{x^\lambda + n^\lambda} \left(\frac{1}{x} \right)^{1-\lambda+\frac{\lambda}{r}} n^{\frac{s(1-\lambda)+\lambda}{r}}$$

is strictly decreasing in $(0, \infty)$. hence we have $\tilde{\omega}_\lambda(n, s) < \omega_\lambda(n, s)$, $n \in \mathbb{N}$. The lemma is proved. ∎

Lemma 2. *For $q > 1$, $\lambda > 0$ and $0 < \varepsilon < \lambda(\sqrt{q} - 1)$, we have*

$$\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\lambda}} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}} du dx = O(1) \quad (\varepsilon \to 0^+). \tag{2.3}$$

Proof. Since $0 < \varepsilon < \frac{\lambda}{\sqrt{q}-1}$, and

$$\begin{aligned} 0 &< \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\lambda}} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}} du dx \\ &< \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\lambda}} \left(\frac{1}{u} \right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}} du dx < \int_1^\infty \frac{1}{x} \int_0^{\frac{1}{x^\lambda}} \left(\frac{1}{u} \right)^{\frac{1}{\sqrt{q}}} du dx \\ &= \frac{\sqrt{q}}{\sqrt{q}-1} \int_1^\infty x^{-1-\lambda+\frac{\lambda}{\sqrt{q}}} dx = \frac{q}{\lambda\left(\sqrt{q}-1\right)^2}. \end{aligned}$$

(2.3) is valid. The lemma is proved. ∎

Theorem 1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n > 0$, $0 < \lambda < \min\{p, q\}$,*

$$0 < \sum_{n=1}^\infty n^{(p-1)(1-\lambda)} a_n^p < \infty, \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{(q-1)(1-\lambda)} b_n^q < \infty$$

then

$$\begin{aligned} 0 &< \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} \\ &< \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=1}^\infty n^{(p-1)(1-\lambda)} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}} \end{aligned} \tag{2.4}$$

where the constant $\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}$ is the best possible. In particular,

(i) *for $\lambda = \frac{1}{2}$, we have*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{\sqrt{m} + \sqrt{n}} < \frac{2\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=1}^\infty n^{\frac{(p-1)}{2}} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{(q-1)}{2}} b_n^q \right\}^{\frac{1}{q}}; \tag{2.5}$$

(ii) *for $p = q = \lambda = 2$, we have*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m^2 + n^2} < \frac{\pi}{2} \left\{ \sum_{n=1}^\infty \frac{1}{n} a_n^2 \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty \frac{1}{n} b_n^2 \right\}^{\frac{1}{2}}. \tag{2.6}$$

Proof. By Hölder's inequality and Lemma 1, since $0 < \lambda \leq \min\{p, q\}$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^\lambda + n^\lambda} \\
= &\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{a_m}{(m^\lambda+n^\lambda)^{\frac{1}{p}}}\left(\frac{m^{\frac{p}{q}}}{n}\right)^{\frac{(1-\lambda)}{p}}\left(\frac{m}{n}\right)^{\frac{\lambda}{pq}}\right] \\
&\qquad \times\left[\frac{b_n}{(m^\lambda+n^\lambda)^{\frac{1}{q}}}\left(\frac{n^{\frac{q}{p}}}{m}\right)^{\frac{(1-\lambda)}{p}}\left(\frac{n}{m}\right)^{\frac{\lambda}{pq}}\right] \\
\leq &\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{m^\lambda+n^\lambda}\left(\frac{m^{\frac{p}{q}}}{n}\right)^{(1-\lambda)}\left(\frac{m}{n}\right)^{\frac{\lambda}{q}}\right\}^{\frac{1}{p}} \\
&\qquad \times\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n^q}{m^\lambda+n^\lambda}\left(\frac{n^{\frac{q}{p}}}{m}\right)^{(1-\lambda)}\left(\frac{n}{m}\right)^{\frac{\lambda}{p}}\right\}^{\frac{1}{q}} \\
= &\left\{\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}\left(\frac{m^{\frac{p}{q}}}{n}\right)^{(1-\lambda)}\left(\frac{m}{n}\right)^{\frac{\lambda}{q}}\right]a_m^p\right\}^{\frac{1}{p}} \\
&\qquad \times\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}\left(\frac{n^{\frac{q}{p}}}{m}\right)^{(1-\lambda)}\left(\frac{n}{m}\right)^{\frac{\lambda}{p}}\right]b_n^q\right\}^{\frac{1}{q}} \\
= &\left\{\sum_{m=1}^{\infty}\tilde{\omega}_\lambda(m,p)\,a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\tilde{\omega}_\lambda(n,q)\,b_n^q\right\}^{\frac{1}{q}} \\
< &\left\{\sum_{n=1}^{\infty}\omega_\lambda(n,p)\,a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega_\lambda(n,q)\,b_n^q\right\}^{\frac{1}{q}} \\
= &\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Inequality (2.4) is valid.

For $\varepsilon \in \left(0, \frac{\lambda}{\sqrt{q}-1}\right)$, setting

$$\tilde{a}_n = \left(\frac{1}{n}\right)^{\frac{[1+\varepsilon+(p-1)(1-\lambda)]}{p}}, \quad \text{and} \quad \tilde{b}_n = \left(\frac{1}{n}\right)^{\frac{[1+\varepsilon+(q-1)(1-\lambda)]}{q}}, \quad (n \in \mathbb{N}),$$

then we have

$$\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}\tilde{a}_n^p = \sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}\tilde{b}_n^q = \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}},$$

and

$$\begin{aligned}\frac{1}{\varepsilon} &= \int_1^{\infty} \frac{1}{x^{1+\varepsilon}}dx < \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \\ &= 1+\sum_{n=2}^{\infty}\frac{1}{n^{1+\varepsilon}} < 1+\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}dx = 1+\frac{1}{\varepsilon}.\end{aligned}$$

Hence we find

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)}\tilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)}\tilde{b}_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\frac{1}{\varepsilon}+O(1)\right\}^{\frac{1}{p}}\left\{\frac{1}{\varepsilon}+O(1)\right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}(1+o(1)) \quad \left(\varepsilon\to 0^+\right).\end{aligned} \tag{2.7}$$

Since $0<\lambda\le\min\{p,q\}$, we have

$$\begin{aligned}\frac{1+\varepsilon+(p-1)(1-\lambda)}{p} &= \frac{\varepsilon}{p}+\left(1-\frac{\lambda}{q}\right)\ge\frac{\varepsilon}{p}>0, \\ \text{and}\quad \frac{1+\varepsilon+(q-1)(1-\lambda)}{q} &\ge \frac{\varepsilon}{q}>0.\end{aligned}$$

We obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}\left(\frac{1}{m}\right)^{\frac{1+\varepsilon+(p-1)(1-\lambda)}{p}}\left(\frac{1}{n}\right)^{\frac{1+\varepsilon+(q-1)(1-\lambda)}{q}} \\ &> \int_1^{\infty}\left[\int_1^{\infty}\frac{1}{x^\lambda+y^\lambda}\left(\frac{1}{x}\right)^{\frac{1+\varepsilon+(p-1)(1-\lambda)}{p}}\left(\frac{1}{y}\right)^{\frac{1+\varepsilon+(q-1)(1-\lambda)}{q}}dy\right]dx \\ &= \frac{1}{\lambda}\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\int_{\frac{1}{x^\lambda}}^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}}dudx \quad (\text{setting } u=\frac{y^\lambda}{x^\lambda}) \\ &= \frac{1}{\lambda}\left[\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}}dudx\right. \\ &\qquad \left. -\int_1^{\infty}\frac{1}{x^{1+\varepsilon}}\int_0^{\frac{1}{x^\lambda}}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}}dudx\right].\end{aligned}$$

Since

$$\int_0^{\infty}\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{q}+\frac{\varepsilon}{q\lambda}}du = \frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}+o(1) \quad \left(\varepsilon\to 0^+\right),$$

by (2.3), we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} &> \frac{1}{\lambda}\left\{\frac{1}{\varepsilon}\left[\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}+o(1)\right]-O(1)\right\} \\ &= \frac{1}{\varepsilon}\cdot\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}(1+o(1)) \quad \left(\varepsilon\to 0^+\right).\end{aligned} \tag{2.8}$$

Suppose that there exists a positive number $K < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$, such that (2.4) is valid by changing $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ to K. In particular, by (2.7) and (2.8), we have

$$\begin{aligned} \frac{1}{\varepsilon} \cdot \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} (1 + o(1)) &< \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m^{\lambda} + n^{\lambda}} \\ &< K \cdot \frac{1}{\varepsilon} (1 + o(1)) \quad \left(\varepsilon \to 0^{+}\right). \end{aligned}$$

It follows that $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \leq K$, which contradicts the fact that $K < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$. Hence the constant $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ in (2.4) is best possible. The theorem is proved. ∎

Theorem 2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, q\}$, $a_n \geq 0$ and*

$$0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{\lambda-1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^p < \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right]^p \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p. \tag{2.9}$$

Inequality (2.9) is equivalent to (2.4). The constant $\left\{ \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right\}^p$ in (2.9) is the best possible. In particular,

(i) *for $\lambda = \frac{1}{2}$, we have*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m} + \sqrt{n}} \right)^p < \left[\frac{2\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \sum_{n=1}^{\infty} n^{\frac{(p-1)}{2}} a_n^p; \tag{2.10}$$

(ii) *for $p = q = \lambda = 2$, we have*

$$\sum_{n=1}^{\infty} n \left(\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right)^2 < \frac{\pi^2}{4} \sum_{n=1}^{\infty} \frac{1}{n} a_n^2. \tag{2.11}$$

Proof. There exists $k_0 \in \mathbb{N}$, such that for $k > k_0$, $\sum_{m=1}^{k} \frac{a_m}{m^{\lambda} + n^{\lambda}} > 0$. Setting

$$b_n(k) = n^{\lambda-1} \left(\sum_{m=1}^{k} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^{p-1} \qquad (k > k_0),$$

by (2.4), we have

$$\begin{aligned} 0 &< \sum_{n=1}^{k} n^{(q-1)(1-\lambda)} b_n^q(k) \\ &= \sum_{n=1}^{k} n^{\lambda-1} \left(\sum_{m=1}^{k} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^p = \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{a_m b_n(k)}{m^{\lambda} + n^{\lambda}} \\ &< \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=1}^{k} n^{(p-1)(1-\lambda)} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{k} n^{(q-1)(1-\lambda)} b_n^q(k) \right\}^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{k} n^{(q-1)(1-\lambda)} b_n^q(k) &= \sum_{n=1}^{k} n^{\lambda-1}\left(\sum_{m=1}^{k}\frac{a_m}{m^\lambda+n^\lambda}\right)^p \\ &< \left[\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=1}^{k} n^{(p-1)(1-\lambda)} a_n^p.\end{aligned}$$

Hence we find

$$0<\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q(\infty)<\infty.$$

By (2.4), we have

$$\begin{aligned} 0 &< \sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^p = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n(\infty)}{m^\lambda+n^\lambda} \\ &< \frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q(\infty)\right\}^{\frac{1}{q}} \\ &= \frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^p\right\}^{\frac{1}{q}}.\end{aligned}$$

It is obvious that inequality (2.9) is valid.

On the other hand, if (2.9) is valid, by Hölder's inequality, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} \\ &= \sum_{n=1}^{\infty}\left[n^{\frac{(\lambda-1)}{p}}\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right]\left[n^{\frac{(1-\lambda)}{p}} b_n\right] \\ &\le \left\{\sum_{n=1}^{\infty} n^{\lambda-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{2.12}$$

By (2.9), we have (2.4). Hence inequality (2.9) is equivalent to (2.4).

If the constant $\left\{\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\right\}^p$ in (2.9) is not best possible, using (2.12), we may get the same result that the constant $\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}$ in (2.4) is not best possible, which is a contradiction. This proves the theorem. ∎

3. The Extended Integral Form

Theorem 3. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $f,g\ge 0$ *and*

$$0<\int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)\,dt<\infty,\ \ 0<\int_0^\infty t^{(q-1)(1-\lambda)} g^q(t)\,dt<\infty,$$

then we have

$$\begin{aligned}(3.1)\quad & \int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x^\lambda + y^\lambda} dxdy \\ & < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)\,dx \right\}^{\frac{1}{q}};\end{aligned}$$

$$(3.2)\quad \int_0^\infty y^{\lambda-1} \left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \right)^p dy < \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)\,dx,$$

where the constant factors $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ *and* $\left\{ \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right\}^p$ *are all the best possible. Inequality (3.1) is equivalent to (3.2).*

Proof. Still by Hölder's inequality, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x^\lambda + y^\lambda} dxdy \\ = & \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(x^\lambda + y^\lambda)^{\frac{1}{p}}} \left(\frac{x^{\frac{p}{q}}}{y} \right)^{\frac{(1-\lambda)}{p}} \left(\frac{x}{y} \right)^{\frac{\lambda}{pq}} \right] \\ & \times \left[\frac{g(y)}{(x^\lambda + y^\lambda)^{\frac{1}{q}}} \left(\frac{y^{\frac{q}{p}}}{x} \right)^{\frac{(1-\lambda)}{p}} \left(\frac{y}{x} \right)^{\frac{\lambda}{pq}} \right] dxdy \\ \le & \left\{ \int_0^\infty \int_0^\infty \frac{f^p(x)}{(x^\lambda + y^\lambda)} \left(\frac{x^{\frac{p}{q}}}{y} \right)^{1-\lambda} \left(\frac{x}{y} \right)^{\frac{\lambda}{q}} dydx \right\}^{\frac{1}{p}} \\ & \times \left\{ \int_0^\infty \int_0^\infty \frac{g^q(x)}{(x^\lambda + y^\lambda)} \left(\frac{y^{\frac{q}{p}}}{x} \right)^{1-\lambda} \left(\frac{y}{x} \right)^{\frac{\lambda}{p}} dydx \right\}^{\frac{1}{q}}. \end{aligned}$$

If (3.3) takes equality, then there exists numbers A and B such that (see [7])

$$\begin{aligned} & A \frac{f^p(x)}{(x^\lambda + y^\lambda)} \left(\frac{x^{\frac{p}{q}}}{y} \right)^{1-\lambda} \left(\frac{x}{y} \right)^{\frac{\lambda}{q}} \\ & \qquad = B \frac{g^q(x)}{(x^\lambda + y^\lambda)} \left(\frac{y^{\frac{q}{p}}}{x} \right)^{1-\lambda} \left(\frac{y}{x} \right)^{\frac{\lambda}{p}} \quad \text{a.e. in } (0,\infty) \times (0,\infty). \end{aligned}$$

It follows that

$$x^{1+(p-1)(1-\lambda)} f^p(x) = y^{1+(q-1)(1-\lambda)} g^q(x) = \text{constant} \quad \text{a.e. in } (0,\infty) \times (0,\infty).,$$

which contradicts the fact that

$$0 < \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)\,dt < \infty.$$

Hence (3.3) takes strict inequality, and in view of (2.1), we have

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x^\lambda + y^\lambda} dxdy < \left\{ \int_0^\infty \omega_\lambda(x,p)\, f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega_\lambda(y,q)\, g^q(y)\, dy \right\}^{\frac{1}{q}}.$$

By Lemma 1, we have (3.1). If the constant $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ in (3.1) is not best possible, then there exists a positive number $k < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ such that (3.1) is valid by changing $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ to k. I particular, for $\varepsilon \in \left(0, \frac{\lambda}{\sqrt{q}-1}\right)$, setting $\tilde{f}(t)$ and $\tilde{g}(t)$ as $\tilde{f}(t) = \tilde{g}(t) = 0$, for $t \in (0,1)$,

$$\tilde{f}(t) = t^{-\frac{[1+\varepsilon+(p-1)(1-\lambda)]}{p}}, \quad \tilde{g}(t) = t^{-\frac{[1+\varepsilon+(q-1)(1-\lambda)]}{q}}, \quad \text{for} \quad t \in [1,\infty),$$

and by (2.8), we have

$$\begin{aligned}
& \frac{1}{\varepsilon} \cdot \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} (1 + o(1)) \\
= \; & \int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\, \tilde{f}(y)}{x^\lambda + y^\lambda} dxdy \\
< \; & k \left\{ \int_0^\infty x^{(p-1)(1-\lambda)} \tilde{f}^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{(q-1)(1-\lambda)} \tilde{g}^q(x)\, dx \right\}^{\frac{1}{q}} \\
= \; & \frac{k}{\varepsilon} \quad \left(\varepsilon \to 0^+\right)
\end{aligned}$$

It follows that $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \le k$, which contradicts the fact that $k < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$. Hence the constant $\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}$ in (3.1) is best possible.

There exists $T_0 > 0$ such that for $T > T_0$, $\int_0^T \frac{f(x)}{x^2+y^2} dx > 0$. Setting

$$g(y,T) = y^{\lambda-1} \int_0^T \frac{f(x)}{x^\lambda + y^\lambda} dx, \quad y \in (0,\infty),$$

by (3.1), we have

$$\begin{aligned}
& \int_0^T y^{(q-1)(1-\lambda)} g^q(y,T)\, dy \\
= \; & \int_0^T y^{\lambda-1} \left(\int_0^T \frac{f(x)}{x^\lambda + y^\lambda} dx \right)^p dy = \int_0^T \int_0^T \frac{f(x)\, g(y,T)}{x^\lambda + y^\lambda} dxdy \\
< \; & \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^T x^{(p-1)(1-\lambda)} f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^T y^{(q-1)(1-\lambda)} g^q(y,T)\, dy \right\}^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T y^{(q-1)(1-\lambda)} g^q(y,T)\, dy &= \int_0^T y^{\lambda-1} \left(\int_0^T \frac{f(x)}{x^\lambda + y^\lambda} dx \right)^p dy \\
&< \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^T x^{(p-1)(1-\lambda)} f^p(x)\, dx.
\end{aligned}$$

Hence we have

$$0 < \int_0^\infty y^{(q-1)(1-\lambda)} g^q (y, \infty)\, dy < \infty.$$

For $T \to \infty$, still by (3.1) we have

$$\int_0^\infty y^{\lambda-1} \left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \right)^p dy < \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p (x)\, dx.$$

Inequality (3.2) is valid. Following the same way of Theorem 1, we may show that inequality (3.1) is equivalent to (3.2). By the equivalence of (3.1) and (3.2), it follows that the constant factor $\left\{ \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \right\}^p$ in (3.2) is best possible. The theorem is proved. ∎

Remark 1. *For $\lambda = 1$, (2.4) changes to (1.1), it follows that (2.4) is a new extension of (1.1). Similarly, (2.9), (3.1) and (3.2) are new extensions of (1.2), (1.3) and (1.4). Since the constant factor in (3.1) is best possible, then (3.1) is more accurate than (1.6).*

References

[1] G.H. HARDY, J.E. LITTLEWOOD and G. POLYA, *Inequalities,* Cambridge: Cambridge University Press, 1952.

[2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities involving functions and their Integrals and Derivatives,* Boston: Kluwer Academic Publishers, 1991.

[3] MINGZHE GAO and BICHENG YANG, On the extended Hilbert's Inequality, *Proceedings of the American Mathematical Society,* **126** (1998), 751-759.

[4] B.G. PACHPATTE, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.,* **226** (1998), 166-179.

[5] BICHENG YANG, On a general Hardy-Hilbert's integral inequality with a best value, *Chin. Ann. of Math.,* **21A** (2000), 401-408.

[6] JICHANG KUANG, On new extensions of Hilbert's integral inequality, *J. Math. Anal. Appl.,* **235** (1999), 608-614.

[7] JICHANG KUANG, *Applied Inequalities,* Changsha: Hunan Education Press, 1992.

Department of Mathematics, Guangdong Education College, Guangzhou, Guangdong 510303, People's Republic of China

E-mail address: bcyang@public.guangzhou.gd.cn