GENERALISED WEIGHTED TRAPEZOIDAL RULES AND RELATIONSHIP TO OSTROWSKI RESULTS

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Abstract. The generalised weighted trapezoidal rule is investigated which involves $f^{(n)}(t)$ of bounded variation. If the weight is taken to be identically unity, then previous results are recaptured. A comparison with weighted Ostrowski results is made and it is demonstrated that if the weight is symmetric about the midpoint over the interval $[a,x)$ and $(x,b]$ then the bounds are the same. In particular, if the weight is unity, then the generalised trapezoidal and Ostrowski results produce the same bounds.

1. Introduction

K.S.K. Iyengar [9], by means of geometrical consideration, has proved the following theorem.

Theorem 1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \le M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \le \frac{M(b-a)^2}{4} - \frac{1}{4M}(f(b)-f(a))^2. \tag{1.1}$$

(See also [16, p. 471 - 474] for related results). Further generalisations were also given by Agarwal and Dragomir [11], and Cerone and Dragomir [13].

In [17], the following generalisation of Theorem 1 is proved analytically.

Theorem 2. *Let $f(x)$ be a differentiable function defined on $[a,b]$ and $|f'(x)| \le M$ for every $x \in (a,b)$. If $p(x)$ is an integrable function on (a,b) such that*

$$0 < c \le p(x) \le \lambda c \quad (\lambda \ge 1,\ x \in [a,b]),$$

then the following inequality holds

$$\left|A(f;p) - \frac{1}{2}(f(a)+f(b))\right| \le \frac{M(b-a)}{2} \cdot \frac{(\lambda+q)\left(1-q^2\right) + 2(\lambda-1)q}{2\lambda(1+q) - (\lambda-1)(1+q^2)}, \tag{1.2}$$

where A and q are defined by

$$A(f;p) = \frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx} \quad \text{and} \quad q = \frac{|f(b)-f(a)|}{M(b-a)}.$$

Cerone and Dragomir [4] also proved the following weighted trapezoidal result.

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Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $[a,b] \subset \mathring{I}$ with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$ be the first moment of $w(\cdot)$ on $[a,b]$. If f' is integrable on $[a,b]$, then the following inequality*

$$\left| \int_a^b w(x) f(x)\, dx - \frac{\nu}{2}[f(a)+f(b)] - m\left(\frac{a+b}{2}\right)[b-a-\nu] \right| \tag{1.3}$$
$$\leq \frac{\nu}{2}(b-a)(S-m) \leq \frac{M-m}{2}\nu(b-a),$$

where S is the slope of the secant on $[a,b]$.

The well-known Ostrowski inequality is given by the following theorem [17],

Theorem 4. *Let f be a differentiable function on $[a,b]$ and let $|f'(x)| \leq M$ on $[a,b]$. Then, for every $x \in [a,b]$,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a) M. \tag{1.4}$$

A weighted version of the Ostrowski inequality (1.4) for Hölder mappings has been given in [6] by Dragomir et al. Other results related to the Ostrowski inequality may be viewed in [18], [21] and a book devoted to Ostrowski type results edited by Dragomir and Rassias [7]. A weighted multidimensional generalisation of Ostrowski's inequality was treated by Milovanović [12].

In a recent thorough article Matić, Pečarić and Ujević [11] obtained weighted $n-$time differentiable Ostrowski type results in terms of a variety of norms. The norms, including when $f^{(n)}$ is of bounded variation, Hölder continuous and differentiable allowing for $f^{(n+1)} \in L_p[a,b]$ in terms of the Lebesgue norms $\|\cdot\|_p$, $p \geq 1$.

It is the intention of the current paper to examine bounds for the generalised weighted trapezoidal functional. Bounds will be provided assuming $f^{(n)}(\cdot)$ to be of bounded variation, absolutely continuous, Lipschitzian and monotonic. Placing restriction on the weight function provides more explicit coarser bounds. This is accomplished through the use of a result due to Karamata [10]. Qi [19] and [20] examines weighted trapezoidal bounds using a Taylor series argument.

Following the presentation of some notation, identities are obtained for our functional of interest in Section 2. Various bounds are developed in Section 3 while in Section 4, the relationship between the trapezoidal and corresponding Ostrowski functionals is investigated. It is demonstrated that the bounds are the same if the weight function is symmetric over the mid-points of the respective intervals $[a,x)$ and $(x,b]$.

2. Some Notation and an Identity

Before proceeding to develop an identity, it is worthwhile to introduce some notation.

Let $w(\cdot)$ be a weight function and suppose that $w : [a,b] \to (0,\infty)$ is integrable on the interval $[a,b]$ and such that

$$0 < \int_a^b w(t)\, dt < \infty.$$

Also, let

$$m_k(c,d;w)=\int_c^d u^k w(u)\,du \tag{2.1}$$

represent the k^{th} moment about the origin of the weight function $w(\cdot)$ over the interval $[c,d]\subseteq[a,b]$. Further, let

$$\begin{aligned} 0 &\leq M_n(a,x;w)=\frac{1}{n!}\int_a^x (u-a)^n w(u)\,du \\ &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-a)^{n-k} m_k(a,x;w) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} 0 &\leq M_n(x,b;w)=\frac{1}{n!}\int_x^b (b-u)^n w(u)\,du \\ &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k m_k(x,b;w). \end{aligned} \tag{2.3}$$

It may be observed that for $x\in[a,b]$

$$M_0(a,b;w)=M_0(a,x;w)+M_0(x,b;w)=\int_a^b w(t)\,dt=m_0(a,b;w)$$

and

$$M_n(a,x;1)=\frac{(x-a)^{n+1}}{(n+1)!},\quad M_n(x,b;1)=\frac{(b-x)^{n+1}}{(n+1)!}. \tag{2.4}$$

We introduce the kernel

$$Q_n(x,t;w):=\begin{cases}\dfrac{1}{(n-1)!}\displaystyle\int_x^t (t-u)^{n-1} w(u)\,du, & n\in\mathbb{N},\\[2ex] w(t), & n=0,\end{cases}\qquad x,t\in[a,b] \tag{2.5}$$

which satisfies

$$\frac{\partial Q_n}{\partial t}=Q_{n-1},\quad n\in\mathbb{N}. \tag{2.6}$$

The kernel may further be written, using (2.2) and (2.3), as

$$Q_n(x,t;w):=\begin{cases}(-1)^n M_{n-1}(t,x;w), & a\leq t\leq x,\\[1ex] M_{n-1}(x,t;w), & x<t\leq b,\end{cases}\qquad n\in\mathbb{N} \tag{2.7}$$

and $Q_0(x,t;w)=w(t)$.

Further, define the functional

$$\begin{aligned} &T_n(a,x,b;f;w) \\ &= \int_a^b w(t) f(t)\,dt-\sum_{k=0}^{n}\left[M_k(a,x;w) f^{(k)}(a)+(-1)^k M_k(x,b;w) f^{(k)}(b)\right] \end{aligned} \tag{2.8}$$

for $f:[a,b]\to\mathbb{R}$, $x\in[a,b]$ and $w(\cdot)$ is a weight function with $M_k(\cdot,\cdot;w)$ as defined by (2.2) and (2.3). The following theorem holds.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ with $a<b$. For $n=0,1,2,\ldots$ let $Q_{n+1}(x,t;w)$ be as given by (2.5). Further, suppose that for some $n\in\mathbb{N}\cup\{0\}$, $f^{(n)}(t)$ exists for $t\in[a,b]$, where $f^{(0)}(t)\equiv f(t)$ then for $f^{(n)}(\cdot)$ of bounded variation the identity*

$$T_n(a,x,b;f;w)=(-1)^{n+1}\int_a^b Q_{n+1}(x,t;w)\,df^{(n)}(t) \tag{2.9}$$

holds where T and Q_{n+1} are as defined by (2.6) and (2.5) respectively.

Proof. Before proceeding with the proof it is worthwhile to firstly note that the solution to the recurrence relation

$$u_n=a_n-u_{n-1}\quad\text{for}\quad n=1,2,\ldots \tag{2.10}$$

is given explicitly as

$$u_n=\sum_{k=1}^{n}(-1)^{n-k}a_k+(-1)^n u_0. \tag{2.11}$$

Now, let

$$I_n=\int_a^b Q_{n+1}(x,t;w)\,df^{(n)}(t) \tag{2.12}$$

then integration by parts of the Riemann-Stieltjes integral gives

$$I_n=Q_{n+1}(x,t;w)\,f^{(n)}(t)\Big]_a^b+\int_a^b\frac{\partial Q_{n+1}}{\partial t}f^{(n)}(t)\,dt \tag{2.13}$$

and so

$$I_n=A_n(a,x,b;w;f)-I_{n-1},\quad n=1,2,\ldots, \tag{2.14}$$

where, to obtain (2.14) from (2.13), we have used

$$A_n(a,x,b;w;f)=Q_{n+1}(x,b;w)\,f^{(n)}(b)-Q_{n+1}(x,a;w)\,f^{(n)}(a) \tag{2.15}$$

and (2.6) and the fact that for $f^{(n-1)}(t)$ differentiable, $df^{(n-1)}(t)=f^{(n)}(t)\,dt$.

To obtain I_0 we may either use integration by parts from (2.12) or equivalently extend the validity of (2.14) to $n=0$, producing

$$I_0=A_0(a,x,b;w;f)-I_{-1},$$

where from (2.12)

$$I_{-1}=\int_a^b Q_0(x,t;w)\,f^{(0)}(t)\,dt$$

and so from (2.5)

$$I_{-1}=\int_a^b w(t)\,f(t)\,dt.$$

That is,

$$I_0=A_0(a,x,b;w;f)-\int_a^b w(t)\,f(t)\,dt. \tag{2.16}$$

The solution of (2.14) on comparison with (2.10) and (2.11) upon using (2.16) is given by

$$I_n=(-1)^{n+1}\int_a^b w(t)\,f(t)\,dt+\sum_{k=0}^{n}(-1)^{n-k}A_k(a,x,b;w;f).$$

Thus,

$$(-1)^{n+1} I_n = \int_a^b w(t) f(t)\,dt - \sum_{k=0}^{n} (-1)^k A_k(a,x,b;w;f) \tag{2.17}$$

where from (2.15) and (2.7)

$$A_k(a,x,b;w;f) = M_k(x,b;w) f^{(k)}(b) - (-1)^{k+1} M_k(a,x;w) f^{(k)}(a)$$

and so from (2.17) the identity (2.9) is procured. ∎

Remark 1. *If $f^{(n)}(t)$ is absolutely continuous on $[a,b]$, then it is differentiable and $df^{(n)}(t) = f^{(n+1)}(t)\,dt$ giving from (2.9) the identity*

$$T_n(a,x,b;f;w) = (-1)^{n+1} \int_a^b Q_{n+1}(x,t;w) f^{(n+1)}(t)\,dt, \tag{2.18}$$

where T_n and Q_n are given by (2.8) and (2.5) or (2.7) respectively.

Remark 2. *It $w(t) \equiv 1$ then from (2.5)*

$$Q_n(x,t;1) = \frac{(t-x)^n}{n!}$$

and from (2.2) and (2.3)

$$M_k(a,x;1) = \frac{(x-a)^{k+1}}{(k+1)!} \quad \text{and} \quad M_k(x,b;1) = \frac{(b-x)^{k+1}}{(k+1)!}.$$

Further, from (2.8) and (2.18)

$$\begin{aligned} T_n(a,x,b;f;1) &= \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \\ &= (-1)^{n+1} \int_a^b \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+1)}(t)\,dt \end{aligned}$$

is the identity obtained in Cerone et al. [5]*, giving the non-weighted $n-$time differentiable generalised trapezoidal identity.*

3. Inequalities for the Generalised Weighted Trapezoidal Rule

The following well known lemmas will prove useful for procuring bounds for a Riemann-Stieltjes integral. They will be stated here for lucidity.

Lemma 1. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous and v is of bounded variation on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left| \int_a^b g(t)\,dv(t) \right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v), \tag{3.1}$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

Lemma 2. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is $L-$Lipschitzian on $[a,b]$. Then*

$$\left| \int_a^b g(t)\,dv(t) \right| \le L \int_a^b |g(t)|\,dt \tag{3.2}$$

with v is $L-$Lipschitzian if it satisfies

$$|v(x) - v(y)| \leq L|x-y|$$

for all $x, y \in [a,b]$.

Lemma 3. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then*

$$\left| \int_a^b g(t)\, dv(t) \right| \leq \int_a^b |g(t)|\, dv(t). \tag{3.3}$$

It should be noted that if v is nonincreasing, then $-v$ is nondecreasing.

Theorem 6. *Let the conditions of Theorem 5 continue to hold so that $f^{(n)}(t)$ is of bounded variation for $t \in [a,b]$. Then we have for all $x \in [a,b]$ and $n \in \mathbb{N} \cup \{0\}$*

$$|T_n(a,x,b;f;w)| \tag{3.4}$$

$$= \begin{cases} [M_n(a,x;w) + M_n(x,b;w) \\ \quad + |M_n(a,x;w) - M_n(x,b;w)|] \times \frac{1}{2} \bigvee_a^b \left(f^{(n)}\right), \\ \\ [M_{n+1}(a,x;w) + M_{n+1}(x,b;w)]\, L, \qquad f^{(n)} \quad \text{is } L-\text{Lipschitzian} \\ \\ M_n(a,x;w) \left[f^{(n)}(x) - f^{(n)}(a)\right] \\ \quad + M_n(x,b;w) \left[f^{(n)}(b) - f^{(n)}(x)\right], \quad f^{(n)} \quad \text{monotonic nondecreasing} \end{cases}$$

where $T_n(a,x,b;f;w)$ is given by (2.8) and $M_n(a,x;w)$, $M_n(x,b;w)$ by (2.2) and (2.3).

Here, by $\bigvee_a^b(h)$ we signify the total variation of $h(t)$ for $t \in [a,b]$. That is $\bigvee_a^b(h) = \int_a^b |dh(t)|$.

Proof. Taking the modulus of identity (2.9) and using Lemma 1, we have

$$\begin{aligned} |T_n(a,x,b;f;w)| &= \left| \int_a^b Q_{n+1}(x,t;w)\, df^{(n)}(t) \right| \\ &\leq \sup_{t \in [a,b]} |Q_{n+1}(x,t;w)| \bigvee_a^b \left(f^{(n)}\right). \end{aligned} \tag{3.5}$$

Now, from (2.5)

$$\begin{aligned} &\sup_{t \in [a,b]} |Q_{n+1}(x,t;w)| \\ &= \frac{1}{n!} \max \left\{ \int_a^x (u-a)^n w(u)\, du, \int_x^b (b-u)^n w(u)\, du \right\} \\ &= \max \{ M_n(a,x;w), M_n(x,b;w) \} \end{aligned} \tag{3.6}$$

and so using the fact that $\max\{X,Y\} = \frac{1}{2}[X + Y + |X-Y|]$ gives the first inequality in (3.4) upon utilising (3.5).

For $f^{(n)}(\cdot)$ $L-$Lipschitzian on $[a,b]$, then from Lemma 2 and (3.5)

$$\begin{aligned}|T_n(a,x,b;f;w)| &= \left|\int_a^b Q_{n+1}(x,t;w)\,df^{(n)}(t)\right| \\ &\leq L\int_a^b |Q_{n+1}(x,t;w)|\,dt.\end{aligned} \tag{3.7}$$

Using the definition (2.5), we may notice that

$$Q_{n+1}(x,t;w)=\begin{cases}\dfrac{(-1)^{n+1}}{n!}\displaystyle\int_t^x (u-t)^n w(u)\,du, & t\in[a,x]\\[2ex] \dfrac{1}{n!}\displaystyle\int_x^t (t-u)^n w(u)\,du, & t\in(x,b]\end{cases} \tag{3.8}$$

and so

$$\begin{aligned}&n!\int_a^b |Q_{n+1}(x,t;w)|\,dt \\ &= \int_a^x\int_t^x (u-t)^n w(u)\,dudt+\int_x^b\int_x^t (t-u)^n w(u)\,dudt.\end{aligned} \tag{3.9}$$

We may simplify the expression on the right by an interchange of the order of integration to give

$$\begin{aligned}\int_a^x\int_t^x (u-t)^n w(u)\,dudt &= \int_a^x w(u)\int_a^u (u-t)^n\,dtdu \\ &= \frac{1}{n+1}\int_a^x (u-a)^{n+1} w(u)\,du\end{aligned}$$

and in a similar fashion

$$\int_x^b\int_x^t (t-u)^n w(u)\,dudt=\frac{1}{n+1}\int_x^b (b-u)^{n+1} w(u)\,du$$

Hence, from (3.9),

$$\int_a^b |Q_{n+1}(x,t;w)|\,dt=M_{n+1}(a,x;w)+M_{n+1}(x,b;w) \tag{3.10}$$

giving the second inequality in (3.4) upon utilising (3.7).

For the final inequality in (3.4), when $f^{(n)}(t)$ is monotonic nondecreasing on $[a,b]$, we use Lemma 3 and thus, from identity (2.9)

$$\begin{aligned}|T_n(a,x,b;f;w)| &= \left|\int_a^b Q_{n+1}(x,t;w)\,df^{(n)}(t)\right| \\ &\leq \int_a^b |Q_{n+1}(x,t;w)|\,df^{(n-1)}(t).\end{aligned} \tag{3.11}$$

Two cases need to be treated. For $n=0$ then

$$\left|T_0(a,x,b;f;w)\right| \le \int_a^b \left|Q_1(x,t;w)\right| df(t),$$

where, from (2.5),

$$\left|Q_1(x,t;w)\right| = \begin{cases} \displaystyle\int_t^x w(u)\,du, & t\in[a,x], \\ \displaystyle\int_x^t w(u)\,du, & t\in(x,b]. \end{cases}$$

Thus,

$$\begin{aligned}
(3.12)\quad & \left|T_0(a,x,b;f;w)\right| \\
\le\ & \int_a^x \left(\int_t^x w(u)\,du\right) df(t) + \int_x^b \left(\int_x^t w(u)\,du\right) df(t) \\
=\ & \left(\int_t^x w(u)\,du\right) f(t)\Bigg]_{t=a}^{x} + \int_a^x w(t) f(t)\,dt \\
& + \left(\int_x^t w(u)\,du\right) f(t)\Bigg]_{t=x}^{b} - \int_x^b w(t) f(t)\,dt \\
=\ & -M_0(a,x;w) f(a) + \int_a^x w(t) f(t)\,dt + M_0(x,b;w) f(b) - \int_x^b w(t) f(t)\,dt \\
\le\ & M_0(a,x;w)\left[f(x)-f(a)\right] + M_0(x,b;w)\left[f(b)-f(x)\right].
\end{aligned}$$

Here we have used the fact that if $g(t)>0$ and $f(t)$ monotonic nondecreasing for $t\in[a,b]$, then

$$\begin{aligned}
(3.13)\qquad \int_a^b g(t) f(t)\,dt &\le f(b)\int_a^b g(t)\,dt \quad \text{and} \\
-\int_a^b g(t) f(t)\,dt &\le -f(a)\int_a^b g(t)\,dt.
\end{aligned}$$

For $n\in\mathbb{N}$ then

$$\begin{aligned}
(3.14)\quad & n!\int_a^b \left|Q_{n+1}(x,t;w)\right| df^{(n)}(t) \\
=\ & \int_a^x \left(\int_t^x (u-t)^n w(u)\,du\right) df^{(n)}(t) + \int_x^b \left(\int_x^t (t-u)^n w(u)\,du\right) df^{(n)}(t) \\
:\ & = A_n + B_n.
\end{aligned}$$

Integration by parts produces on using the Leibnitz rule,

$$\begin{aligned} A_n &= \left(\int_t^x (u-t)^n w(u)\,du\right) f^{(n)}(t)\Bigg]_{t=a}^{x} + n\int_a^x \left(\int_t^x (u-t)^{n-1} w(u)\,du\right) f^{(n)}(t)\,dt \\ &= -n!M_n(a,x;w)\, f^{(n)}(a) + n\int_a^x \left(\int_t^x (u-t)^{n-1} w(u)\,du\right) f^{(n)}(t)\,dt \\ &\leq n!M_n(a,x;w)\left[f^{(n)}(x) - f^{(n)}(a)\right] \end{aligned}$$

and

$$\begin{aligned} B_n &= \left(\int_x^t (t-u)^n w(u)\,du\right) f^{(n)}(t)\Bigg]_{t=x}^{b} - n\int_x^b \left(\int_x^t (t-u)^{n-1} w(u)\,du\right) f^{(n)}(t)\,dt \\ &= n!M_n(x,b;w)\, f^{(n)}(b) - n\int_x^b \left(\int_x^t (t-u)^{n-1} w(u)\,du\right) f^{(n)}(t)\,dt \\ &\leq n!M_n(x,b;w)\left[f^{(n)}(b) - f^{(n)}(x)\right], \end{aligned}$$

where we have used the monotonicity of $f^{(n)}(\cdot)$ via (3.13) together with (3.9) and (3.10) to obtain the upper bounds.

Substituting A_n and B_n into (3.14) and (3.11) and further recognising that it subsumes the result for $n=0$ as given by (3.12), then the last inequality in (3.4) results. ∎

Remark 3. *For the monotonic nondecreasing result in (3.4) a tighter bound could have been obtained if the result (3.13) were not used. This, however, would have produced a more cumbersome bound. The $n=0$ case may have been accommodated in the general n case since, as given in (2.5), $Q_0(x,t;w) = w(t)$ and since $w(t) > 0$ then $|Q_0(x,t;w)| = w(t)$.*

The following theorem gives bounds on $|T_n(a,x,b;f;w)|$ in terms of the Lebesgue norms of $f^{(n+1)}(t)$.

Theorem 7. *Let the general conditions of Theorem 5 hold and further, let $f^{(n)}(t)$ be absolutely continuous for $t \in [a,b]$, then*

$$(3.15)\qquad |T_n(a,x,b;f;w)|$$

$$\leq \begin{cases} [M_{n+1}(a,x;w) + M_{n+1}(x,b;w)]\left\|f^{(n+1)}\right\|_\infty, & f^{(n+1)} \in L_\infty[a,b]; \\ \|Q_n(x,\cdot;w)\|_q \left\|f^{(n+1)}\right\|_p, & f^{(n+1)} \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ [M_n(a,x;w) + M_n(x,b;w) & \\ \quad + |M_n(a,x;w) - M_n(x,b;w)|]\dfrac{\left\|f^{(n+1)}\right\|_1}{2}, & f^{(n+1)} \in L_1[a,b], \end{cases}$$

where $M_n(a,x;w)$ and $M_n(x,b;w)$ are as given by (2.2) and (2.3), $Q_n(a,x,b;w)$ is defined in (2.5) and $T_n(a,x,b;w)$ by (2.8).

Further, $\|\cdot\|_p$ *signify the usual Lebesgue norms where*

$$\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)| \quad for \quad h \in L_\infty[a,b],$$

and

$$\|h\|_p := \left(\int_a^b |h(t)|^p \, dt \right)^{\frac{1}{p}} \quad for \quad h \in L_p[a,b],\ 1 \le p < \infty.$$

Proof. From the identity (2.18) we have on using properties of the modulus and integral,

$$|T_n(a,x,b;w)| \le \int_a^b \left| Q_{n+1}(x,t;w) f^{(n+1)}(t) \right| dt. \tag{3.16}$$

Now, for $f^{(n+1)} \in L_\infty[a,b]$

$$\int_a^b \left| Q_{n+1}(x,t;w) f^{(n+1)}(t) \right| dt \le \left\| f^{(n+1)} \right\|_\infty \int_a^b |Q_{n+1}(x,t;w)| \, dt,$$

which upon using (3.10) produces the first inequality in (3.15).

For the second bound we use Hölder's integral inequality in (3.16) to give

$$\begin{aligned} \int_a^b \left| Q_{n+1}(x,t;w) f^{(n+1)}(t) \right| dt &\le \left(\int_a^b |Q_{n+1}(x,t;w)|^q \, dt \right)^{\frac{1}{q}} \left(\int_a^b \left| f^{(n+1)}(t) \right|^p dt \right)^{\frac{1}{p}} \\ &= \|Q_n(x,\cdot;w)\|_q \left\| f^{(n+1)} \right\|_p, \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The final inequality in (3.15) is obtained from (3.16) to give

$$\int_a^b \left| Q_{n+1}(x,t;w) f^{(n+1)}(t) \right| dt \le \sup_{t\in[a,b]} |Q_{n+1}(x,t;w)| \int_a^b \left| f^{(n+1)}(t) \right| dt,$$

where we may use (3.6) and a property of the $\max\{X,Y\}$ to obtain the stated result. ∎

Remark 4. *If we take the weight function $w(t) \equiv 1$ then the results of Cerone et al.* [5] *involving the generalised trapezoidal rule and $n-$time differentiable functions is recaptured.*

A question that needs to be asked is can we choose the parameter x in such a way that the bound is minimized? The following lemma examines such an issue.

Lemma 4. *Let*

$$\begin{aligned} & 2\phi_n(a,x,b;w) \\ = \ & [M_n(a,x;w) + M_n(x,b;w) + |M_n(a,x;w) - M_n(x,b;w)|] \end{aligned} \tag{3.17}$$

and

$$\Psi_{n+1}(a,x,b;w) = M_{n+1}(a,x;w) + M_{n+1}(x,b;w). \tag{3.18}$$

Then

$$\phi_n^*(a,\tilde{x},b;w) = \min_{x\in[a,b]} \phi_n(a,x,b;w) = \frac{M_n(a,\tilde{x};w) + M_n(\tilde{x},b;w)}{2}, \tag{3.19}$$

where $\tilde{x}$ is the solution of $M_n(a,x;w) = M_n(x,b;w)$.

Further,

$$\Psi^*_{n+1}\left(a,\frac{a+b}{2},b;w\right) = \min_{x\in[a,b]}\Psi_{n+1}(a,x,b;w) = M_{n+1}\left(a,\frac{a+b}{2};w\right)+M_{n+1}\left(\frac{a+b}{2},b;w\right), \tag{3.20}$$

where $w(\cdot)$ is a positive weight function and $x\in[a,b]$, with $M_n(a,x;w)$ and $M_n(x,b;w)$ are as defined in (2.2) and (2.3).

Proof. The functions $M_n(a,x;w)$ and $M_n(x,b;w)$ are both positive with $M_n(a,x;w)$ increasing and $M_n(x,b;w)$ decreasing in $x\in[a,b]$. Thus, the minimum is attained when $|M_n(a,x;w)-M_n(x,b;w)|=0$ giving the result as stated.

Now, $\Psi_{n+1}(a,x;w)\ge 0$ and

$$\Psi_{n+1}(a,a;w)=\Psi_{n+1}(b,b;w)=M_n(a,b;w).$$

Also, for $w(x)>0$

$$\Psi'_{n+1}(a,x,b;w)=\left[(x-a)^n-(b-x)^n\right]\frac{w(x)}{n!}$$

and so $\Psi'_{n+1}(a,a,b;w)<0$, $\Psi'_{n+1}(a,b,b;w)>0$ bringing us to the conclusion that $\Psi_n(a,x,b;w)$ is convex in x. Since $w(x)>0$, the minimum is attained when $x=\frac{a+b}{2}$, making $\Psi'_{n+1}\left(a,\frac{a+b}{2},b;w\right)=0$. ∎

The following lemma obtains some coarser bounds which may prove to be more useful in practice. It involves obtaining bounds on

$$\Psi_{n+1}(a,x,b;w)=\|Q_{n+1}(x,\cdot;w)\|_1=M_{n+1}(a,x;w)+M_{n+1}(x,b;w). \tag{3.21}$$

Lemma 5. *Let $w(t)$ be a weight function defined on $[a,b]$ and $x\in[a,b]$, then*

$$\begin{aligned}|\Psi_{n+1}(a,x,b;w)| &= \|Q_{n+1}(x,\cdot;w)\|_1 \\ &\le \begin{cases} C(1)\|w\|_\infty, & w\in L_\infty[a,b]; \\ C^{\frac{1}{2}}(q)\|w\|_p, & w\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\nu^{n+1}}{(n+1)!}\|w\|_1, & w\in L_1[a,b], \end{cases}\end{aligned} \tag{3.22}$$

where

$$C(q)=\frac{(x-a)^{q(n+1)+1}+(b-x)^{q(n+1)+1}}{q(n+1)+1}$$

and

$$\nu=\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|.$$

Proof. From the definitions (2.2) and (2.3) it may be noticed that $\Psi_{n+1}(a,x,b;w)$ from (3.21) may be expressed as

$$\begin{aligned}\Psi_{n+1}(a,x,b;w) &= \|Q_{n+1}(a,x,b;w)\|_1 \\ &= \frac{1}{(n+1)!}\int_a^b \kappa^{n+1}(a,x,b;u)\,w(u)\,du,\end{aligned} \tag{3.23}$$

where

$$\kappa(a,x,b;u)=\begin{cases} u-a, & u\in[a,x], \\ \\ b-u, & u\in(x,b]. \end{cases} \tag{3.24}$$

Now,

$$(n+1)!\left|\Psi_{n+1}(a,x,b;w)\right|\leq\int_a^b\left|\kappa^{n+1}(a,x,b;u)\,w(u)\right|du$$

and so for $w\in L_p[a,b]$, $1<p<\infty$ then

$$\begin{aligned}&\int_a^b\left|\kappa^{n+1}(a,x,b;u)\,w(u)\right|du \\ \leq\ & \left(\int_a^b\kappa^{q(n+1)}(a,x,b;u)\,du\right)^{\frac{1}{q}}\left(\int_a^b w^p(u)\,du\right)^{\frac{1}{p}}.\end{aligned} \tag{3.25}$$

Explicitly,

$$\begin{aligned}\left(\int_a^b\kappa^{q(n+1)}(a,x,b;u)\,du\right)^{\frac{1}{q}} &= \left[\int_a^x(u-a)^{q(n+1)}\,du+\int_x^b(b-u)^{q(n+1)}\,du\right]^{\frac{1}{q}} \\ &= \left[\frac{(x-a)^{q(n+1)+1}+(b-x)^{q(n+1)+1}}{q(n+1)+1}\right]^{\frac{1}{q}}\end{aligned}$$

which together with (3.25) gives the second inequality (3.22).

For $w\in L_\infty[a,b]$, then

$$\int_a^b\left|\kappa^{n+1}(a,x,b;u)\,w(u)\right|du\leq C(1)\left\|w\right\|_\infty$$

the first inequality in (3.22).

Finally, for $w\in L_1[a,b]$, then

$$\int_a^b\left|\kappa^{n+1}(a,x,b;u)\,w(u)\right|du\leq\left[\sup_{x\in[a,b]}\kappa(a,x,b;u)\right]^{n+1}\left\|w\right\|_1$$

where

$$\sup_{x\in[a,b]}\kappa(a,x,b;u)=\max\{x-a,b-x\}=\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|=\nu.$$

∎

Karamata [10] proved the following theorem.

Theorem 8. *Let $g,w:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and suppose $m\leq g(t)\leq M$ and $0<c\leq w(t)\leq\lambda c$ for $t\in[a,b]$ and some constants m,M,c and λ. If G and $A(g,w)$ are defined as*

$$G:=\frac{1}{b-a}\int_a^b g(t)\,dt\quad\text{and}\quad A(g,w):=\frac{\int_a^b g(t)\,w(t)\,dt}{\int_a^b w(t)\,dt} \tag{3.26}$$

then

$$\frac{\lambda m(M-G)+M(G-m)}{\lambda(M-G)+(G-m)}\leq A(g,w)\leq\frac{m(M-G)+\lambda M(G-m)}{(M-G)+\lambda(G-m)}. \tag{3.27}$$

Using the above theorem of Karamata, the third inequality in (3.22) may be improved.

If we associate $\kappa^{n+1}(a,x,b;w)$, as defined by (3.24), with $g(t)$ above, then

$$0 \le \kappa(a,x,b;u) \le \nu = \max\{x-a, b-x\} = \frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|$$

and

$$G = \frac{C(1)}{b-a} = \frac{1}{b-a}\int_a^b \kappa^{n+1}(a,x,b;u)\,du.$$

Hence, from (3.23)

$$\begin{aligned}(n+1)!\Psi_{n+1}(a,x,b;w) &= \|Q_{n+1}(x,\cdot;w)\|_1 \\ &\le \frac{\lambda\nu^{n+1}C(1)\|w\|_1}{(b-a)\nu^{n+1} - C(1) + \lambda C(1)} \\ &\le \nu^{n+1}\|w\|_1.\end{aligned}$$

The last inequality follows from the fact that

$$C(1) = \int_a^b \kappa^{n+1}(a,x,b;u)\,du \le \nu^{n+1}(b-a).$$

4. Comparison of Ostrowski and Trapezoidal Results

The generalised weighted trapezoid kernel $Q_n(x,t;w)$ defined by (2.5) and (2.7) is a mapping $Q_n(\cdot,\cdot;w) : [a,b]^2 \to \mathbb{R}$, $a,b \in \mathbb{R}$, $a<b$ where

$$(4.1)\qquad Q_n(x,t;w) := \begin{cases} \dfrac{(-1)^n}{(n-1)!}\displaystyle\int_t^x (u-t)^{n-1} w(u)\,du, & t\in[a,x], \quad n\in\mathbb{N} \\ \dfrac{(-1)^n}{(n-1)!}\displaystyle\int_x^t (t-u)^{n-1} w(u)\,du, & t\in(x,b], \quad n\in\mathbb{N} \\ w(t), & \qquad\qquad n=0. \end{cases}$$

Here $w(t)$ is a weight function with properties as ascribed earlier in the paper. An identity relating the generalised weighted trapezoid functional $T_n(a,x,b;f;w)$ as defined by (2.8) is given by (2.9).

In our notation define the weighted Ostrowski functional $\Theta(a,x,b;f;w)$ by

$$(4.2)\qquad \Theta_n(a,x,b;f;w) := \int_a^b w(t)f(t)\,dt - \sum_{k=0}^{n} E_k(a,x,b;w)f^{(k)}(x),$$

where

$$(4.3)\qquad E_k(a,x,b;w) = \frac{1}{k!}\int_t^x (u-x)^k w(u)\,du.$$

Let the kernel $K_n(x,t;w)$ be such that $K_n(\cdot,\cdot,w):[a,b]^2\to\mathbb{R}$, $a,b\in\mathbb{R}$, $a<b$ and $w(\cdot)$ a given weight function, then

$$K_n(x,t;w):=\begin{cases}\dfrac{1}{(n-1)!}\displaystyle\int_a^t (t-u)^{n-1}w(u)\,du, & t\in[a,x), \quad n\in\mathbb{N}\\[2ex] 0 & t=x\\[2ex] \dfrac{(-1)^n}{(n-1)!}\displaystyle\int_t^b (u-t)^{n-1}w(u)\,du, & t\in(x,b], \quad n\in\mathbb{N}\\[2ex] w(t), & x,t\in[a,b].\end{cases} \tag{4.4}$$

Matić et al. [11] show that for $f^{(n)}(\cdot)$ continuous and of bounded variation on $[a,b]$ then for $x\in[a,b]$

$$\Theta_n(a,x,b;f;w)=(-1)^{n+1}\int_a^b \kappa^{n+1}(x,t;w)\,df^{(n)}(t). \tag{4.5}$$

Further, for $f^{(n)}(\cdot)$ absolutely continuous on $[a,b]$ then $df^{(n)}(t)=f^{(n+1)}(t)\,dt$, giving the identity

$$\Theta_n(a,x,b;f;w)=(-1)^{n+1}\int_a^b \kappa^{n+1}(x,t;w)\,f^{(n+1)}(t)\,dt \tag{4.6}$$

from (4.5).

The bounds for $|T(a,x,b;f;w)|$ and $|\Theta(a,x,b;f;w)|$ depend on the behaviour of $|Q_{n+1}(x,t;w)|$ and $|K_{n+1}(x,t;w)|$ respectively.

The following lemma gives sufficient conditions for the bounds on $|T(a,x,b;f;w)|$ and $|\Theta(a,x,b;f;w)|$ to be equal.

Lemma 6. *For $w:[a,b]\to(0,\infty)$, $\int_a^b w(t)\,dt<\infty$ and $w(t)$ symmetric about the respective midpoints for $t\in[a,x)$ and $t\in(x,b]$ then,*

$$|K_n(x,t;w)|=\begin{cases}|Q_n(x,a+x-t;w)|, & t\in[a,x),\\[1ex] 0 & t=x\\[1ex] |Q_n(x,x+b-t;w)|, & t\in(x,b].\end{cases} \tag{4.7}$$

An interchange of Q_n and K_n in (4.7) is valid under the same conditions.

Proof. Consider for $t\in[a,x)$

$$\begin{aligned}(n-1)!\,|Q_n(x,a+x-t;w)| &= \int_{a+x-t}^x [u-(a+x-t)]^{n-1}w(u)\,du\\ &= \int_a^t (t-v)^{n+1}w(a+x-v)\,dv\\ &= (n-1)!\,|K_n(x,t;w)|,\end{aligned}$$

provided $w(a+x-v)=w(v)$, that is, $w\left(\frac{a+x}{2}-z\right)=w\left(\frac{a+x}{2}+z\right)$ for $z\in[a,x)$. A similar argument gives the result for $t\in(x,b]$. We note that if t varies from c to d, then $T=c+d-t$ varies from d to c. Thus the interchange of K_n and Q_n in (4.7) is valid and the equivalent expression to (4.7) holds. ∎

Remark 5. *A consequence of Lemma 6 is that if* $w(t) \equiv \alpha$*, a constant, then the Lebesgue norms giving the bounds for* $|T(a,x,b;f;w)|$ *and* $|\Theta(a,x,b;f;w)|$ *are equal. In particular, the unweighted case* $w(t) = \alpha = 1$ *produce the same bounds as shown in Cerone* [2]*. Weights such as*

$$w_n(t) = \begin{cases} \left(t - \frac{a+x}{2}\right)^{2n}, & t \in [a,x) \\ 0, & t = x \\ \left(t - \frac{x+b}{2}\right)^{2n}, & t \in (x,b], \quad n \in \mathbb{N} \end{cases}$$

and

$$W(t) = \begin{cases} \left|t - \frac{a+x}{2}\right|, & t \in [a,x) \\ 0, & t = x \\ \left|t - \frac{x+b}{2}\right|, & t \in (x,b], \end{cases}$$

would produce the same bounds for the trapezoidal and Ostrowski functions defined by (2.9) and (4.2).

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